INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (1)

Why is the normal mode analysis insufficient ?

1. The NMA is relevant only for the long-time (asymptotic) behavior of the disturbance field.

2. If large transient amplification of disturbances is possible then small perturbations of the flow may quickly rise up to the magnitude which is sufficient to trigger nonlinear effects. Then the flow will probably evolve to a new (laminar or chaotic) state, even though the Reynolds number Re_L is subcritical and all normal modes of the undisturbed motion are nominally stable.

So what are we going to do?

1. We will calculate the form of the “most dangerous” (or optimal) disturbances, i.e. such that give rise to the largest possible amplification.

2. We will carry out the parametric study of the transient growth phenomenon.

3. We will look at the structures in the disturbance velocity field accompanying the transient growth process.

INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (2)

Transient energy growth in the referential Poiseuille flow

Disturbance velocity field: $\mathbf{v}(t,x,y,z) = [g_u, g_v, g_w](t,y)\exp[i(\delta x + \beta z)] + C.C.$

Initial/boundary value problem (linear theory) is formulated as follows:

OS Eq. $\{\partial_t(\partial_{yy} - k^2) + i\beta[W_0(\partial_{yy} - k^2) - D^2 W_0] - \frac{1}{Re}(\partial_{yy} - k^2)^2\} g_v = 0,$

Sq Eq. $[\partial_t + i\beta W_0 - \frac{1}{Re}(\partial_{yy} - k^2)]\theta = \underline{\underline{i\,\delta D W_0 g_v}},$ $\longleftarrow$ **forcing term !**

Bound.C. $g_v(t,\pm 1) = \partial_y g_v(t,\pm 1) = \theta(t,\pm 1) = 0,$

Init. C. $g_v(0,y) = g_v^0(y)$, $\theta(0,y) = \theta^0(y)$,

where $W_0(y) = 1 - y^2$, $k = \sqrt{\delta^2 + \beta^2}$.

The O-S equation is homogeneous but the Sq equations contains the forcing term! Even if the vertical velocity decay in time it could make the vertical vorticity grow for some time. The growth of the vertical vorticity (if appears) will produce large disturbances of the horizontal velocity components.

INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (3)

The following special case in very instructive …

Assume that $\beta = 0$. Then all OS and Sq modes are attenuated and not moving, i.e. all eigenvalues are purely imaginary and have negative real parts. Consider the initial conditions:

$$g_v(0,y) = \hat{G}_v(y) \text{ (selected O-S mode)}, \quad \theta(0,y) \equiv 0.$$

The solution of the initial/boundary value problem can be written as

$$g_v(t,y) = \hat{G}_v(y)\exp(-\hat{\zeta}^{OS} t) \ ,$$

$$\theta(t,y) = i\,\delta \sum_{(j)} [\int_{-1}^{1} DW_0(y)\,\hat{G}_v(y)\,\Xi_j^*(y)\,dy] \frac{\exp(-\hat{\zeta}^{OS} t) - \exp(-\zeta_j^{Sq} t)}{\hat{\zeta}^{OS} - \zeta_j^{Sq}}\, \Xi_j(y),$$

where:

$\hat{\sigma}^{OS} = -i\,\hat{\zeta}^{OS}$ - purely imaginary eigenvalue of the selected O-S mode, $\hat{\zeta}^{OS} > 0$,

$\sigma_j^{Sq} = -i\,\zeta_j^{Sq}$ - purely imaginary eigenvalue of all Sq modes, $\zeta_j^{Sq} > 0$.

INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (4)

Special case continued …

For short times we can expand in the power series and get

$$\theta(t,y) = i\delta t \sum_{(j)} \left[\int_{-1}^{1} DW_0(y)\hat{G}_v(y)\Xi_j^*(y)\,dy \right] \Xi_j(y) + O(t^2) \approx$$

$$\approx i\delta DW_0(y) g_v(0,y)\, t + O(t^2)$$

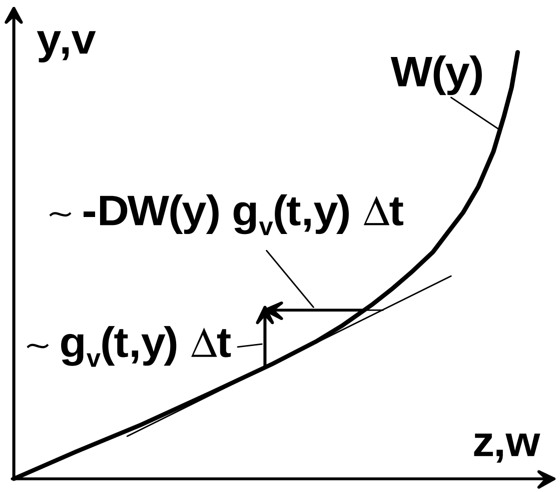
For $\beta = 0$ we have $\theta(t,y) = -i\delta g_w(t,y)$ hence $g_w(t,y) \approx \underbrace{-DW_0(y) g_v(0,y)\, t}_{\text{TRANSIENT LINEAR GROWTH}} + O(t^2).$

For a short time interval we have

$$g_w(t+\Delta t, y) \approx -DW_0(y) g_v(t,y)\Delta t,$$

which is interpreted physically as the

LIFT-UP EFFECT

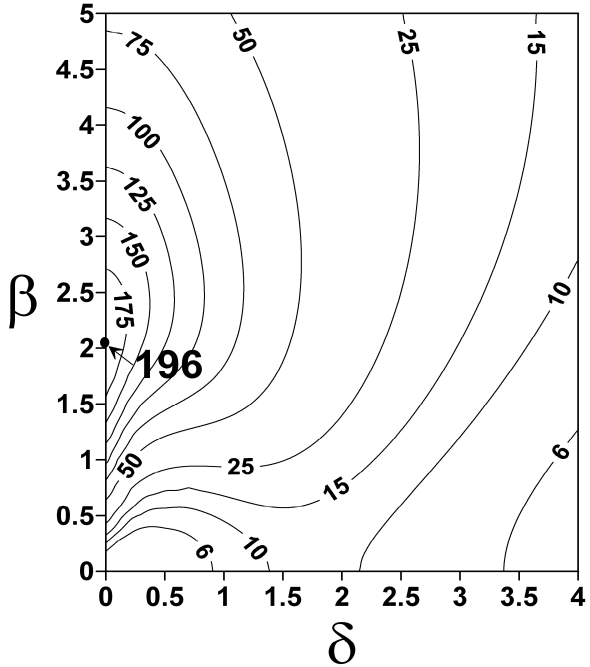


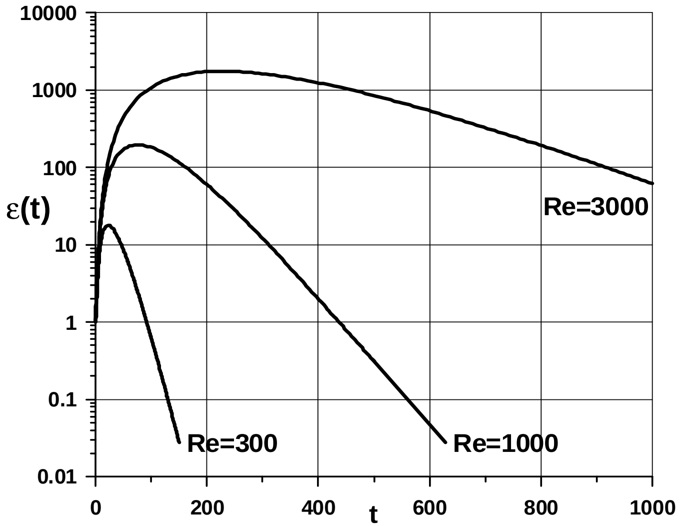
INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (5)

Essential quantitative results for the Poiseuille flow …

The disturbance energy norm $\varepsilon(t) = \frac{1}{2k^2}\int_{-1}^{1}\{|\partial_y g_v|^2 + k^2|g_v|^2 + |\theta|^2\}dy$

We choose $t = \tau$ and look for the IC $\{g_v^M(y), \theta^M(y)\}$ such that $\varepsilon(0) = 1$ and $\varepsilon(\tau)$ is the largest. The maximal energy depends on τ, i.e. $\varepsilon = \varepsilon_M(\tau)$. If Re < Re_L then $\varepsilon_M(\tau)$ attains the maximum ε_{opt} at $\tau = \tau_{opt}$. The corresponding initial conditions are called **the optimal disturbances**.





Left: the map of ε_{opt} as the function of δ and β for Re=1000.

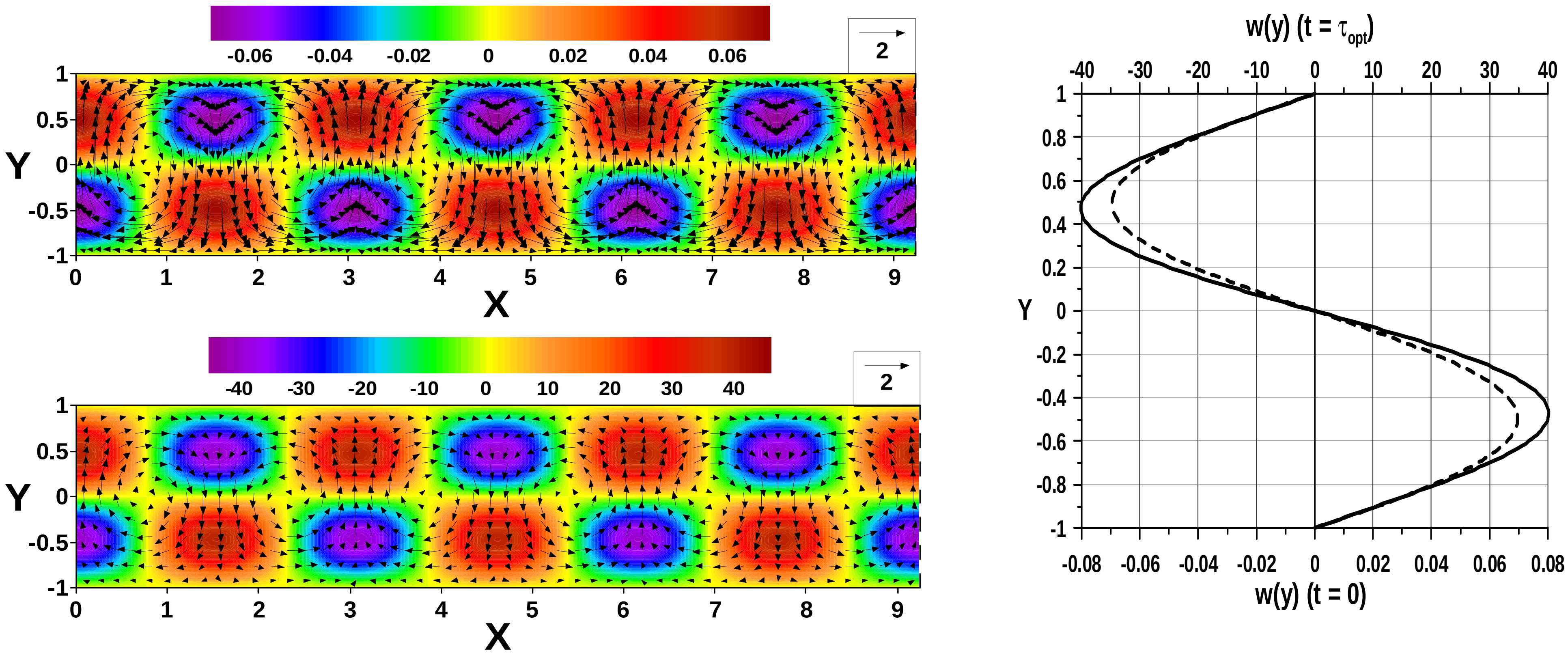
Right: the energy of the disturbances evolving in time from the optimal IC, computed for different Reynolds numbers. The wave numbers are δ=0 and β=2.04 (optimal for all values of Re).

Scaling of the optimum with the Reynolds number:

$$\varepsilon_{opt} \approx 1.96 \cdot 10^{-4}\, Re^2 \quad , \quad \tau_{opt} \approx 0.076 \cdot Re$$

INITIAL VALUE PROBLEM AND TRANSIENT GROWTH OF DISTURBANCE ENERGY (6)

Structure of the optimal disturbances in the Poiseuille flow



Left: the spanwise structure of the velocity field at the time t=0 (top) and t=τ_{opt}=76 (bottom), computed for Re=1000, δ=0 and β=2.04. The corresponding energy growth factor is 196.

Right: The velocity profile of the "streamwise streak" at the time t=0 and t=τ_{opt}=76. The streamwise velocity is amplified about 600 times (!). All parameters like for the contour maps.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (1)

Formulation of the problem

Finite dimensional approximation of the disturbance dynamics

$$\boldsymbol{M}\tfrac{\mathrm{d}}{\mathrm{dt}}\mathbf{z}+\boldsymbol{L}\,\mathbf{z}=\mathbf{0} \quad , \quad \boldsymbol{B}\,\boldsymbol{z}=\mathbf{0} \qquad \Rightarrow \qquad \tfrac{\mathrm{d}}{\mathrm{d\,t}}\mathbf{z}=-\mathrm{i}\,\boldsymbol{A}\,\mathbf{z} \quad , \quad \boldsymbol{A}=-\mathrm{i}\begin{bmatrix}\boldsymbol{M}\\ \boldsymbol{B}\end{bmatrix}^{-1}\cdot\begin{bmatrix}\boldsymbol{L}\\ \boldsymbol{0}\end{bmatrix},$$

\+ the initial condition $\mathbf{z}(0)=\mathbf{z}_0$.

The energy norm $\varepsilon(\mathrm{t},\mathbf{z}_0)=\langle\mathbf{z}(\mathrm{t},\mathbf{z}_0),\mathbf{E}\,\mathbf{z}(\mathrm{t},\mathbf{z}_0)\rangle\equiv\mathbf{z}^{\mathrm{H}}(\mathrm{t},\mathbf{z}_0)\mathbf{E}\,\mathbf{z}(\mathrm{t},\mathbf{z}_0)$, **E** – HPD matrix

The ”most dangerous” disturbances can be found by solving the following problem :

Having a fixed time instant $\mathrm{t}=\tau>0$, find such vector $\mathbf{z}_0^{\mathrm{M}}$ that $\varepsilon(0,\mathbf{z}_0^{\mathrm{M}})=1$ (normalization) and $\varepsilon_{\mathrm{M}}=\varepsilon(\tau,\mathbf{z}_0^{\mathrm{M}})$ is the largest possible. By solving this problem for different values of τ , we obtain the envelope function of the maximal energy growth (amplification) $\varepsilon=\varepsilon_{\mathrm{M}}(\tau)$.

Optimal initial disturbances (only for Re < Re_{L}): we maximize $\varepsilon_{\mathrm{M}}(\tau)$ with respect to τ and obtain $\mathbf{z}_0^{\mathrm{OPT}}$, τ_{OPT} and $\varepsilon_{\mathrm{OPT}}$.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (2)

Solution of the constrained optimization problem

The solution of IVP can be written as $\mathbf{z}(\tau, \mathbf{z}_0) = \exp(-\mathrm{i}\boldsymbol{A}\tau)\,\mathbf{z}_0$.

Assume (for simplicity) that $\boldsymbol{A}$ diagonalizable $\boldsymbol{A} = \boldsymbol{V}\,\Sigma\boldsymbol{V}^{-1} \Rightarrow \exp(-\mathrm{i}\boldsymbol{A}\tau) = \boldsymbol{V}\exp(-\mathrm{i}\boldsymbol{\Sigma}\tau)\boldsymbol{V}^{-1}$,
where $\exp(-\mathrm{i}\boldsymbol{\Sigma}\tau) = \mathrm{diag}\{\exp(-\mathrm{i}\sigma_1\tau),..,\exp(-\mathrm{i}\sigma_{M_G}\tau)\}$ and M_G = dim $\boldsymbol{A}$.

Then $\mathbf{z}(\tau, \mathbf{z}_0) = \boldsymbol{V}\exp(-\mathrm{i}\boldsymbol{\Sigma}\tau)\boldsymbol{V}^{-1}\mathbf{z}_0 = \boldsymbol{V}\exp(-\mathrm{i}\boldsymbol{\Sigma}\tau)\mathbf{s}_0$, where $\mathbf{z}_0 = \boldsymbol{V}\,\mathbf{s}_0$.

The energy norm can be expressed in terms of the vector $\mathbf{s}_0$:

$\tilde{\varepsilon}(\tau, \mathbf{s}_0) = \left\langle \mathbf{s}_0, \tilde{\mathbf{E}}(\tau)\mathbf{s}_0 \right\rangle$, where $\tilde{\mathbf{E}}(\tau) = \exp(\mathrm{i}\boldsymbol{\Sigma}^*\tau)\boldsymbol{V}^{\mathrm{H}}\mathbf{E}\boldsymbol{V}\exp(-\mathrm{i}\boldsymbol{\Sigma}\tau)$ (HPD matrix).

The normalization condition is $\tilde{\varepsilon}(0, \mathbf{s}_0) = \left\langle \mathbf{s}_0, \tilde{\mathbf{E}}(0)\mathbf{s}_0 \right\rangle = 1$.

The extended functional $\boldsymbol{J}[\mathbf{s}_0] = \tilde{\varepsilon}(\tau, \mathbf{s}_0) - \mu[\tilde{\varepsilon}(0, \mathbf{s}_0) - 1]$. Stationary points of the functional $\boldsymbol{J}$ are the eigenvectors of the generalized HPD eigenvalue problem $\tilde{\mathbf{E}}(\tau)\mathbf{s}_0 = \mu\tilde{\mathbf{E}}(0)\mathbf{s}_0$.

Solution to the optimization problem is the eigenvector $\mathbf{s}_0^{\mathrm{M}}$ with the largest eigenvalue $\mu_{\mathrm{M}} = \tilde{\varepsilon}(\tau, \mathbf{s}_0^{\mathrm{M}}) \equiv \varepsilon_{\mathrm{M}}(\tau)$ and $\mathbf{z}_0^{\mathrm{M}} = \boldsymbol{V}\,\mathbf{s}_0^{\mathrm{M}}$.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (3)

Numerical analysis of the transient energy growth

General remarks:

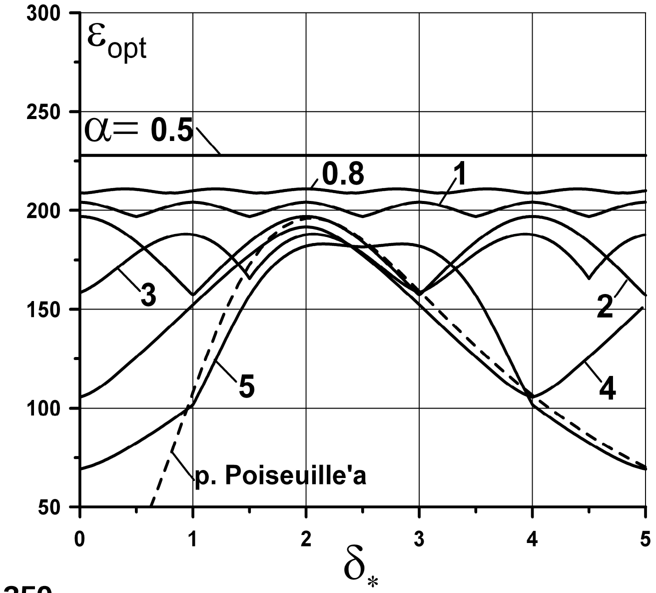
(1) Each eigenmode in the wavy channel can be treated as originating from a certain eigensolution of the referential Poiseuille flow with the wave vector $\boldsymbol{\kappa}_m = [\kappa_x, 0, \kappa_z] = [\delta_* + m\alpha, 0, \beta]$, m = -$M_S$,..,0,..,$M_S$.Thus, in contrast to the Poiseuille flow we will now look for initial disturbances, which are built with many spanwise Fourier harmonics.

(2) Theoretically (i.e. with M_S approaching infinity) the ensemble of all eigenmodes is periodic with the Floquet number δ_*, and the period is equal to the wave number α. Thus, it is actually sufficient to consider δ_* in the range $[0,\alpha)$ or $[-\alpha/2, \alpha/2)$.

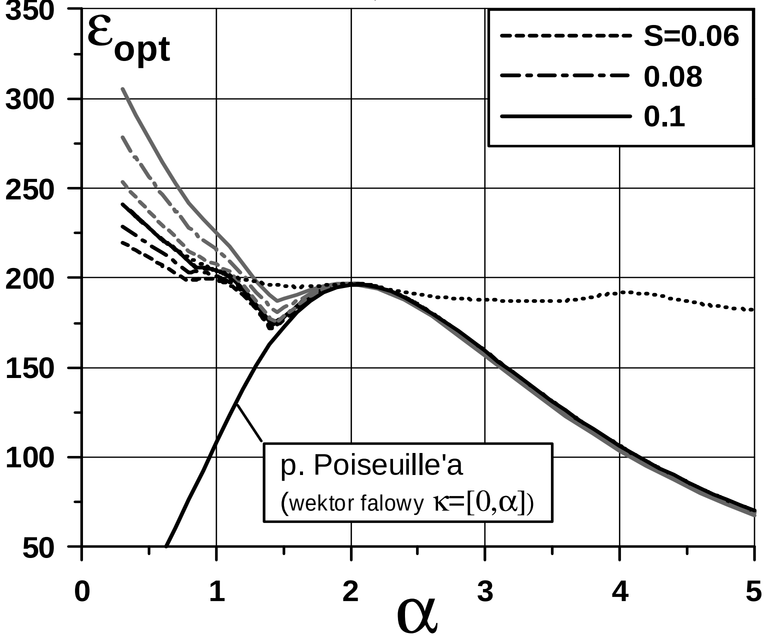
(3) Numerical calculations show that the largest transient growth is observed for the disturbances which are streamwise independent ($\beta = 0$), exactly like in the case of the Poiseuille flow.

(4) If $\beta = \delta_* = 0$ then the simplified description of the disturbance dynamics must be slightly changed. In such case the 0-th Fourier modes of the vertical velocity and vorticity components are zero. Thus, they cannot be used to evaluate the 0-th Fourier modes of the streamwise and spanwise components of the velocity – the latter are included into governing equations as the explicit unknowns. Also, in order to obtain well-posed problem two additional constrains must be imposed on the disturbance field. Here we assume that the mean values of the streamwise and spanwise pressure gradients remain unchanged.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (4)

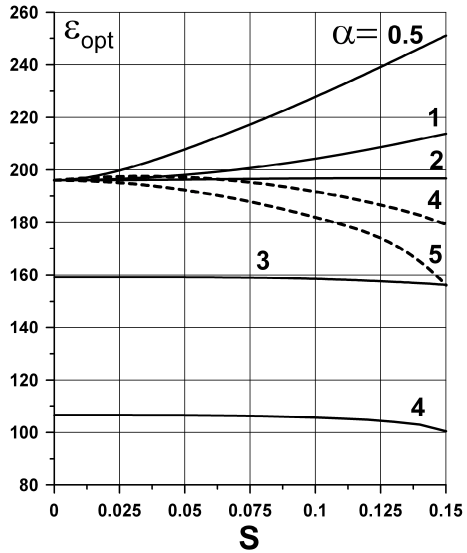


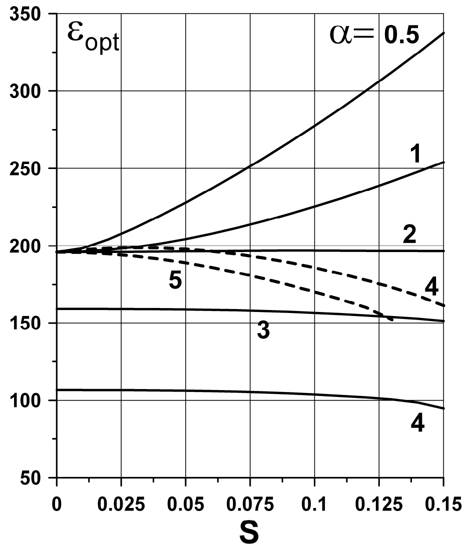
Optimal energy amplification as the function of the Floquet parameter δ_*, computed for different values of the wall wave number α (Re = 1000, S=0.1). Dashed line corresponds to the referential flow with the same Reynolds number and the spanwise wane number equal $\delta = \delta_*$.



Optimal energy amplification as the function of the wall wave number α computed for different amplitudes of the corrugation of the bottom (black lines) and both walls (symmetric case, gray lines). Reynolds number Re=1000, the Floquet parameter $\delta_* = 0$. The dot line shows the value of the optimal amplification obtained with the Floquet parameter adjusted in such a way that the Fourier mode with the spanwise wave number equal 2 is always present.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (5)

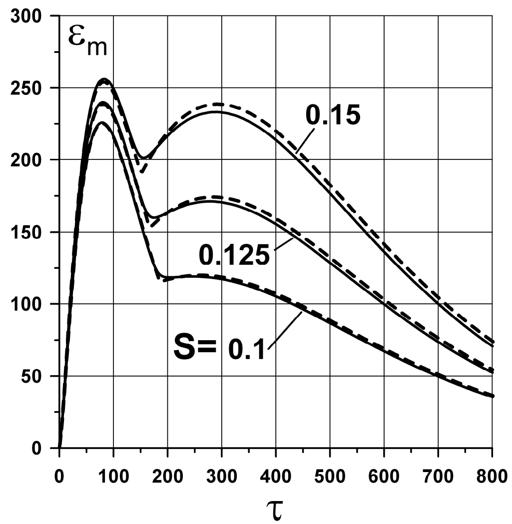


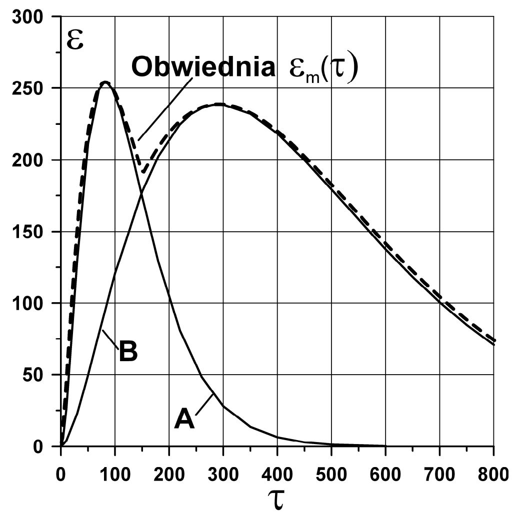


OEA as the function of the amplitude S computed for 1-sided (left) and 2-sided symmetric (right) wall corrugation with different geometric period. The Reynolds number Re=1000. The Floquet parameter δ_* is either 0 (continuous lines) or 2 (dashed lines).

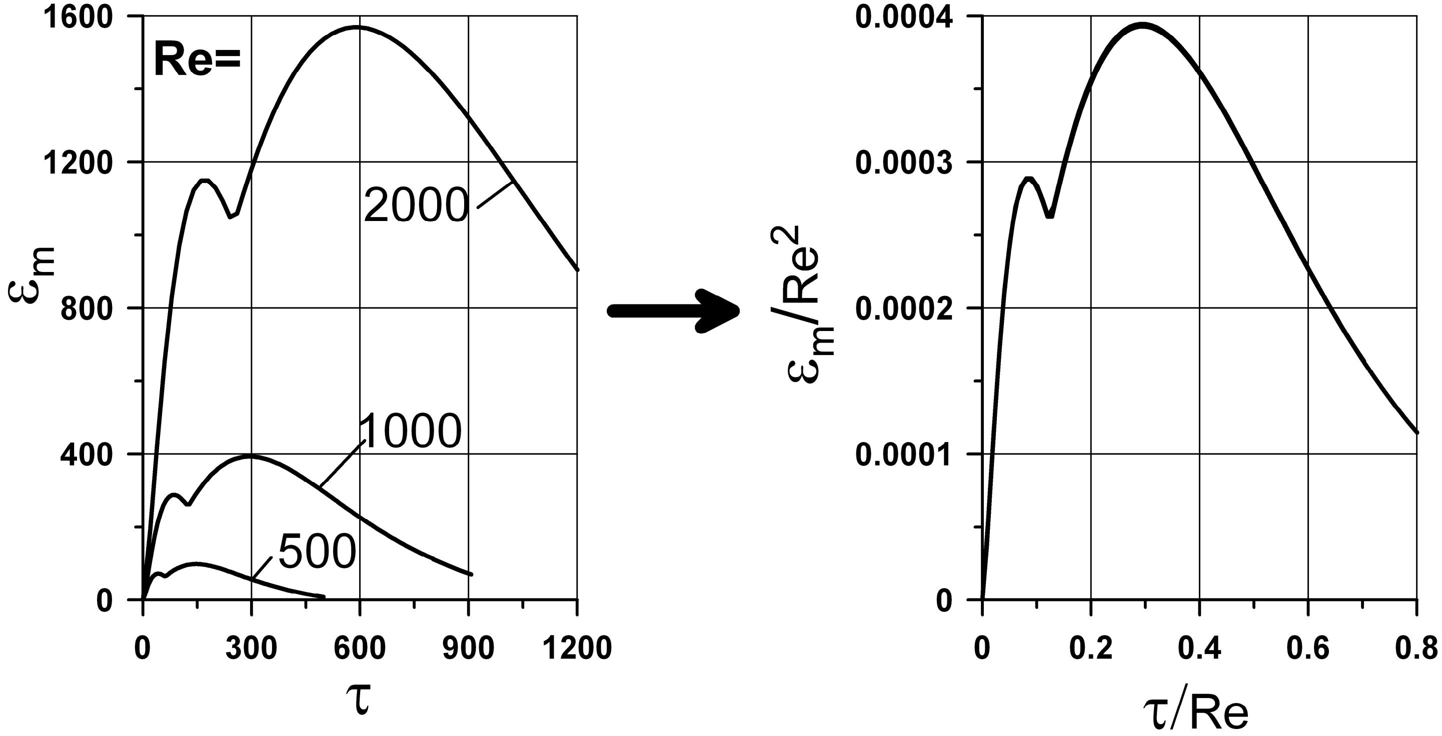
Left: the maximal energy amplification as the function of the time instant τ computed for Re=1000, α=1, δ_*=0 and different values of the amplitude S (2-sided symmetric wavy walls)

Right: time histories of the disturbance energy computed for the initial conditions corresponding to local maxima of the envelope function $\varepsilon = \varepsilon_M(\tau)$.





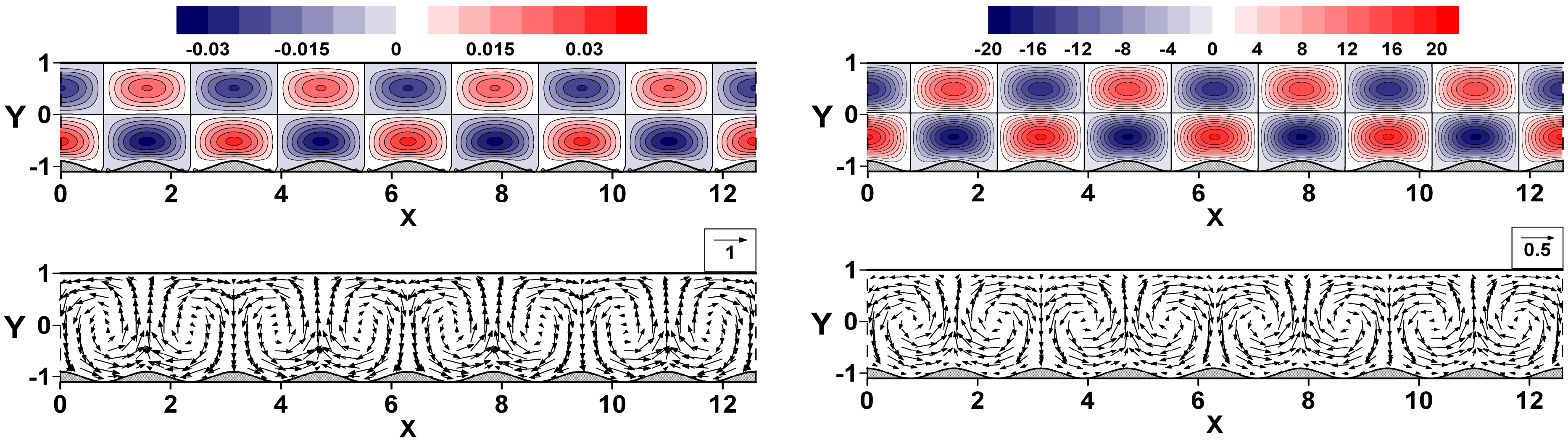
TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (6)



Demonstration of the universal scaling of the maximal energy growth envelope function $\varepsilon = \varepsilon_M(\tau)$ with respect to the Reynolds number (essentially the same as for the Poiseuille flow) (2-sided symmetric wall corrugation with S=0.2, α=1 and δ_*=0).

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (7)

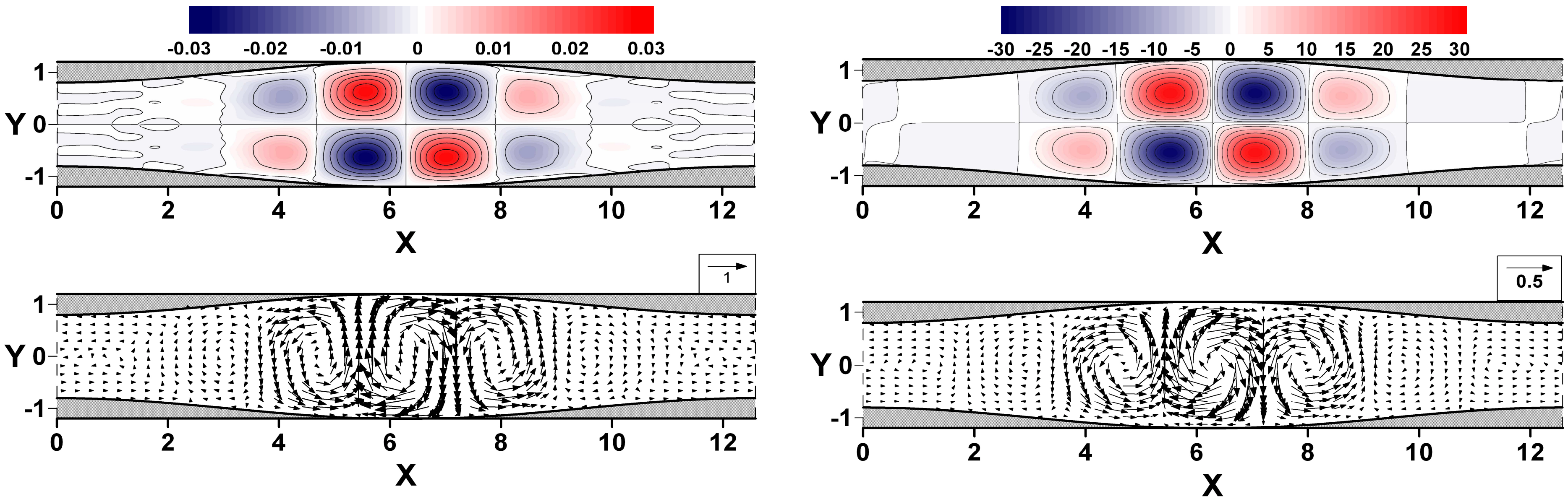
Flow structures for the first maximum (lift-up effect) …



Spanwise structure of the velocity field corresponding to the optimal initial disturbance in the wavy channel with S=0.1 and α=4, computed for Re=1000, δ_*=2 and β=0. The spanwise period of the velocity field is 2 times larger that the wall period. Left: optimal initial condition (t=0). Right: the state obtained for the optimal time t≈75. The optimal amplification of disturbance energy is about 192. The streamwise component of the disturbance velocity is amplified about 600 times.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (8)

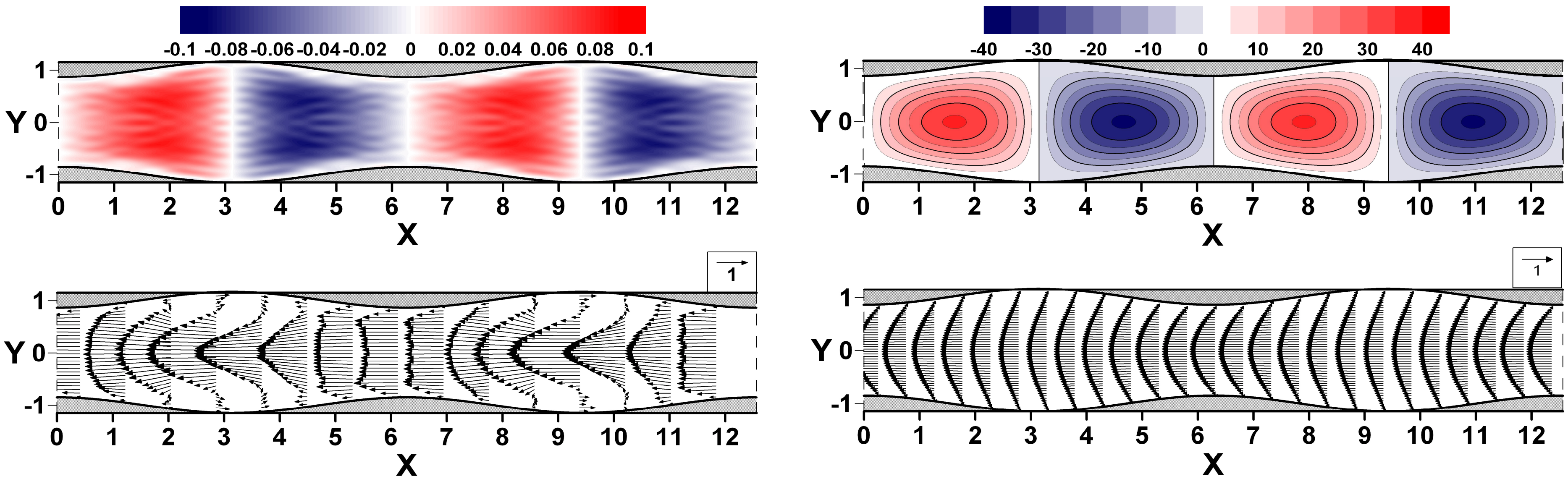
Flow structures for the first maximum (lift-up effect) continued …



As above but this time the wall wave number is α=0.5 and the amplitude S=0.2. The energy amplification attains the optimal value of 408 at the time instant t ≈ 96. The streamwise velocity is amplified about 1000 times (!)

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (9)

Flow structures for the second maximum …



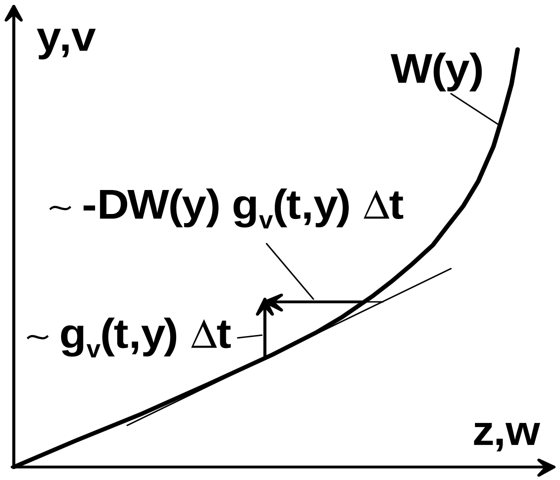
Spanwise structure of the disturbance velocity field corresponding to the second maximum, calculated for S=0.15, α=1, Re=1000 and $\delta_*=\beta=0$.

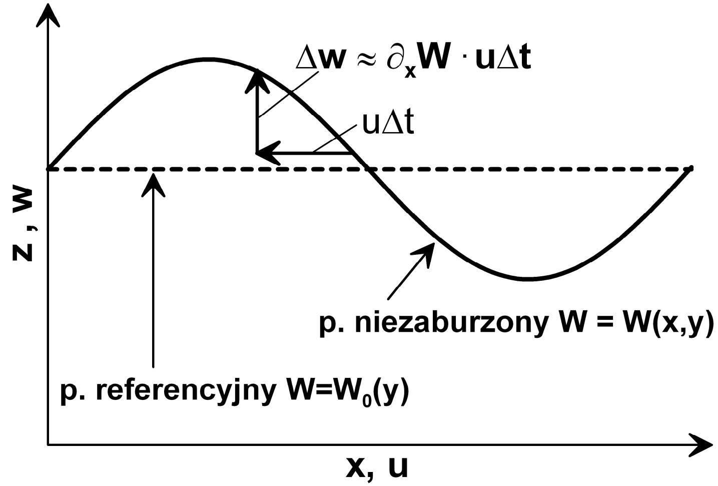
Left: optimal initial condition (t=0). Note the lack of the streamwise vortices, very small vertical velocity component and dominating cross-flow with nonzero volumetric rate.

Right: the state obtained for the optimal time t≈292. The optimal amplification of disturbance energy is about 239. The strong streamwise streak structure emerges but quite different then before. The cross-flow still exists but is slowly attenuated. The streamwise component of the disturbance velocity is amplified about 450 times.

TRANSIENT ENERGY GROWTH IN THE WAVY CHANNEL (10)

Mechanism of the transient energy growth





LEFT: The **lift-up mechanism**: slowly attenuated spanwise-periodic array of streamwise-oriented vortices generate vertical disturbances which are transformed into the streamwise streaks with temporarily large amplitude. This mechanism of energy growth is common for all 2D parallel or nearly parallel flows (Poiseuille, Couette, Blasius, Falkner-Skan, etc.).

RIGHT: The **“push-aside” mechanism**: the slowly attenuated cross-flow disturbances (with nonzero net flux) interact with the spanwise-modulated velocity of the basic flow and also generate streamwise streaks, however with different structure then in the case of the lift-up effect.

From the mathematical viewpoint, the phenomenon of the transient growth appears due to strong nonorthogonality of a few least attenuated normal modes (the evolutionary linear operator is nonnormal).

THE SUMMARY OF THE ANALYSIS OF THE TRANSIENT ENERGY GROWTH

1. Large transient growth of the disturbance energy is possible providing that Fourier modes with the spanwise wave number equal or close to 2 are present in the disturbance field. Such situation is natural for the wall corrugation with the large period where "most active" Fourier modes will always appear as superhamonics.

2. Two "modes" of the transient growth may appear: the short-time mode and long-time mode. The first one is driven by the lift-up mechanism and characteristic time of the extreme energy amplification is less that 100. The long-time mode (appears for larger amplitudes S and $\alpha \approx 1$) is driven by the push-aside effect, which is not present in the referential flow. The characteristic time scale of the push-aside effect is roughly 3 times larger that for the lift-up effect.

3. The transient velocity structures generated by lift-up and push-aside mechanism differ in the form of the streamwise streaks and in the character of the transversal motion. The lift-up effect is connected to the presence of the slowly decaying vortices occupying the whole volume of fluid (good for mixing!). However, in the flow structures generated by the push-aside effect the disturbed motion is predominantly horizontal (the vertical velocity is very small), thus mixing in such disturbance field may be less effective.

4. The streamwise streaks are known to be very unstable structures – later development of secondary instabilities should occur and lead to even more complicated 3D velocity field with enhanced mixing properties. Interesting effects may also expected due to the secondary instability of the cross-flow in the disturbance velocity field generated by the push-aside effect. Much further research is required in this respect, including high-resolution DNS.