Saturation of Estimates for the Maximum Enstrophy Growth in a Hydrodynamic System as an Optimal Control Problem

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Agenda

**Background**
- Regularity Problem for Navier–Stokes Equation
- Enstrophy Estimates

**Saturation of Estimates as Optimization Problem**
- Instantaneous Estimates
- Finite-Time Estimates
- Burgers Problem

**Results**
- Optimal Solutions for Wavenumber $m = 1$
- Envelopes & Power Laws
- Solutions for Other Initial Guesses $m = 2, 3, \ldots$
Navier–Stokes equation \((\Omega = [0, L]^d, \; d = 2, 3)\)

\[
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p - \nu \Delta \mathbf{v} &= 0, \quad \text{in } \Omega \times (0, T] \\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega \times (0, T] \\
\text{Initial Condition} &\quad \text{on } \Gamma \times (0, T] \\
\text{Boundary Condition (periodic)} &\quad \text{in } \Omega \text{ at } t = 0
\end{aligned}
\]

- **2D Case**
  - Existence Theory Complete — smooth and unique solutions exist for arbitrary times and arbitrarily large data

- **3D Case**
  - Weak solutions (possibly nonsmooth) exist for arbitrary times
  - Classical (smooth) solutions (possibly nonsmooth) exist for finite times only
  - Possibility of “blow–up” (finite–time singularity formation)
  - One of the Clay Institute “Millennium Problems” ($1\text{M!}$)
    
    http://www.claymath.org/millennium/Navier-Stokes_Equations
What is known? — Available Estimates

► A Key Quantity — Enstrophy

\[ \mathcal{E}(t) \triangleq \int_\Omega |\nabla \times \mathbf{v}|^2 d\Omega \quad (= \| \nabla \mathbf{v} \|_2^2) \]

► Smoothness of Solutions \iff Bounded Enstrophy
(Foias & Temam, 1989)

\[ \max_{t \in [0, T]} \mathcal{E}(t) < \infty \quad ??? \]

► Can estimate \( \frac{d\mathcal{E}(t)}{dt} \) using the momentum equation, Sobolev’s embeddings, Young and Cauchy–Schwartz inequalities, ...

► Remark: incompressibility not used in these estimates ....
2D Case:

\[ \frac{d\mathcal{E}(t)}{dt} \leq \frac{C^2}{\nu} \mathcal{E}(t)^2 \]

- Gronwall’s lemma and energy equation yield \( \forall t \mathcal{E}(t) < \infty \)
- smooth solutions exist for all times

3D Case:

\[ \frac{d\mathcal{E}(t)}{dt} \leq \frac{27C^2}{128\nu^3} \mathcal{E}(t)^3 \]

- corresponding estimate not available ....
- upper bound on \( \mathcal{E}(t) \) blows up in finite time

\[ \mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4 \frac{C\mathcal{E}(0)^2}{\nu^3} t}} \]

- singularity in finite time cannot be ruled out!
Saturation of Estimates as Optimization Problem

Results

Instantaneous Estimates

Finite-Time Estimates

Burgers Problem

Problem of Lu & Doering (2008), I

Can we actually find solutions which “saturate” a given estimate?

Estimate \( \frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3 \) at a fixed instant of time \( t \)

\[
\max_{v \in H^1(\Omega), \nabla \cdot v = 0} \frac{d\mathcal{E}(t)}{dt}
\]

subject to \( \mathcal{E}(t) = \mathcal{E}_0 \)

where

\[
\frac{d\mathcal{E}(t)}{dt} = -\nu \|\Delta v\|_2^2 + \int_\Omega v \cdot \nabla v \cdot \Delta v \, d\Omega
\]

\( \mathcal{E}_0 \) is a parameter

Solution using a gradient–based descent method
Problem of Lu & Doering (2008), II

\[ \left[ \frac{d\mathcal{E}(t)}{dt} \right]_{max} = 8.97 \times 10^{-4} \mathcal{E}_0^{2.997} \]
How about solutions which saturate \( \frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^3 \) over a finite time window \([0, T]\)?

\[
\begin{align*}
\text{max}_{v \in H^1(\Omega), \nabla \cdot v = 0} \left[ \text{max}_{t \in [0, T]} \mathcal{E}(t) \right] \\
\text{subject to } \mathcal{E}(t) = \mathcal{E}_0
\end{align*}
\]

where

\( \mathcal{E}(t) = \int_0^t \frac{d\mathcal{E}(\tau)}{d\tau} d\tau + \mathcal{E}_0 \)

- \( \mathcal{E}_0 \) is a parameter
- \( \text{max}_{t \in [0, T]} \mathcal{E}(t) \) nondifferentiable w.r.t initial condition
  \( \implies \) non–smooth optimization problem

In principle doable, but will try something simpler first ...
Burgers equation ($\Omega = [0, 1], \, u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in } \Omega$$

$$u(x) = \phi(x) \quad \text{at } t = 0$$

Periodic B.C.

Enstrophy: $\mathcal{E}(t) = \frac{1}{2} \int_0^1 |u_x(x, t)|^2 \, dx$

Solutions smooth for all times

Questions of sharpness of enstrophy estimates still relevant

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{3}{2} \left( \frac{1}{\pi^2 \nu} \right)^{1/3} \mathcal{E}(t)^{5/3}$$

Best available finite-time estimate

$$\max_{t \in [0, T]} \mathcal{E}(t) \leq \left[ \mathcal{E}_0^{1/3} + \left( \frac{L}{4} \right)^2 \left( \frac{1}{\pi^2 \nu} \right)^{4/3} \mathcal{E}_0 \right]^3$$

$\mathcal{E}_0 \rightarrow \infty \quad \Rightarrow \quad C_2 \mathcal{E}_0^3$
“Small” Problem of Lu & Doering (2008), I

- Estimate $\frac{d\mathcal{E}(t)}{dt} \leq c\mathcal{E}(t)^{5/3}$ at a fixed instant of time $t$

$$\max_{u \in H^1(\Omega)} \frac{d\mathcal{E}(t)}{dt}$$

subject to $\mathcal{E}(t) = \mathcal{E}_0$

where

- $\frac{d\mathcal{E}(t)}{dt} = -\nu \left\| \frac{\partial^2 u}{\partial x^2} \right\|^2 + \frac{1}{2} \int_0^1 \left( \frac{\partial u}{\partial x} \right)^3 d\Omega$

- $\mathcal{E}_0$ is a parameter

- Solution (maximizing field) found analytically! (in terms of elliptic integrals and Jacobi elliptic functions)
“Small” Problem of Lu & Doering (2008), II

\[
\left[ \frac{d\mathcal{E}(t)}{dt} \right]_{\text{max}} = 0.2476 \mathcal{E}_0^{5/3} \nu^{1/3}
\]

instantaneous estimate is sharp

finite–time estimate far from saturated

\[
\max_{t \in [0, T]} \mathcal{E}(t) \leq C \mathcal{E}_0^{1.048}
\]
Finite–Time Optimization Problem (I)

Statement

\[
\max_{u \in H^1(\Omega)} E(T) \\
\text{subject to } E(t) = E_0
\]

\(T, E_0\) — parameters

Optimality Condition

\[
\forall \phi' \in H^1 \quad J'_\lambda(\phi; \phi') = - \int_0^1 \frac{\partial^2 u}{\partial x^2} \bigg|_{t=T} u' \bigg|_{t=T} \, dx - \lambda \int_0^1 \frac{\partial^2 \phi}{\partial x^2} \bigg|_{t=0} u' \bigg|_{t=0} \, dx
\]
Finite–Time Optimization Problem (II)

- **Gradient Descent**

\[ \phi^{(n+1)} = \phi^{(n)} - \tau^{(n)} \nabla J(\phi^{(n)}), \quad n = 1, \ldots, \]

\[ \phi^{(0)} = \phi_0, \]

where \( \nabla J \) determined from *adjoint system* via \( H^1 \) Sobolev preconditioning

\[- \frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - \nu \frac{\partial^2 u^*}{\partial x^2} = 0 \quad \text{in} \ \Omega \]

\[ u^*(x) = - \frac{\partial^2 u}{\partial x^2}(x) \quad \text{at} \ t = T \]

Periodic B.C.

- **Step size** \( \tau^{(n)} \) found via *arc minimization*
Two parameters: \( T, E_0 \quad (\nu = 10^{-3}) \)

Optimal initial conditions corresponding to initial guess with wavenumber \( m = 1 \) (local maximizers)

Fixed \( E_0 = 10^3 \), different \( T \)

Fixed \( T = 0.0316 \), different \( E_0 \)
Background
Saturation of Estimates as Optimization Problem

Results
Optimal Solutions for Wavenumber $m = 1$

Envelopes & Power Laws
Solutions for Other Initial Guesses $m = 2, 3, \ldots$

$\max_{t \in [0, T]} E(t) \sim C E_0^{-0.5}$

$\max_{t \in [0, T]} E(t) \sim C E_0^{1.5}$

D. Ayala & B. Protas
Maximum Enstrophy Growth in Burgers Equation
Sol’ns found with initial guesses $\phi^{(m)}(x) = \sin(2\pi mx), \ m = 1, 2, \ldots$

$m = 1, \mathcal{E}_0 = 10^3$

$m = 2, \mathcal{E}_0 = 10^3$

Change of variables leaving Burgers equation invariant ($L \in \mathbb{Z}^+$):

$x = L\xi, \ (x \in [0, 1], \ \xi \in [0, 1/L]), \quad \tau = t/L^2$

$\nu(\tau, \xi) = Lu(x(\xi), t(\tau)), \quad \mathcal{E}_\nu(\tau) = L^4\mathcal{E}_u \left( \frac{t}{L^2} \right)$
Solutions for $m = 1$ and $m = 2$, after rescaling

Using initial guess:
\[
\phi^{(0)}(x) = \sin(2\pi mx), \quad m = 1, \text{ or } m = 2
\]
\[
\phi^{(0)}(x) = \epsilon \sin(2\pi mx) + (1 - \epsilon) \sin(2\pi nx), \quad m \neq n, \quad \epsilon > 0
\]

All local maximizers with $m = 2, 3, \ldots$ are rescaled copies of the $m = 1$ maximizer
Location of Singularities in $\mathbb{C}$ from the Fourier spectrum

$$|\hat{u}_k| \sim C|k|^{-\alpha}e^{iz^*} \quad \text{as} \quad k \to \infty$$

Analyticity strip for a meromorphic function
Saturation of Estimates as Optimization Problem

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$\mathcal{F}\{z^*(t)\}$.

$\mathcal{E}(t)$

- **RED** — instantaneously optimal (Lu & Doering, 2008)
- **BOLD BLUE** — finite–time optimal ($T = 0.1$)
- **DASHED BLUE** — finite–time optimal ($T = 1$)
Summary & Conclusions

- Some evidence that optimizers found are in fact *global*
- Exponents in $\max_{t \in [0, T]} \mathcal{E}(t) = C\mathcal{E}_0^{\alpha}$ as $\mathcal{E}_0 \to \infty$

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<th>theoretical estimate</th>
<th>optimal (instantaneous)</th>
<th>optimal (finite–time)</th>
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<td></td>
<td>[Lu &amp; Doering, 2008]</td>
<td>[present study]</td>
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- More rapid enstrophy build–up in finite–time optimizers than in instantaneous optimizers
- Theoretical estimate *not sharp* $\implies$ finite–time optimizers offer insights re: refinements required (work in progress)

- Finite–time maximizers saturate Poincaré’s inequality (largest kinetic energy for a given enstrophy)
- Future work: Navier–Stokes 2D, 3D...