LINEAR AND NONLINEAR ANALYSIS OF NUMERICAL METHOD FOR SIMULATION OF CONVECTION FLOWS

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Introduction

At last time many researchers studied thermal Rayleigh-Benard convection using numerical simulation. As rule, they used spectral methods with periodic boundary conditions. In numerical simulations were derived secondary stationary, periodic, quasiperiodic and stochastic regimes [1,2]. Some authors performed 2-D and 3-D simulations for high supercriticality with free [3,4] and rigid [5,6] boundary conditions on the horizontal plates.

So far the full simulation of time-dependent three-dimensional convection is a very complex problem demanding large resources. The reasons are: 1. The existence of rapidly increasing and rapidly decreasing of harmonics in linear approach (at r = Ra/Racr = 1000 and Pr = 1 one of harmonics increases as $e^{682\,t}$). 2. The necessity of conformity in linear approach of spectral characteristics of differential problem and numerical method [7]. 3. The necessity of calculations on the enough big time of order of several of thermal diffusion time with enough big number of degrees of freedom.

As a rule, results of simulations of convection with free horizontal boundaries have a bad agreement with experimental data and results of simulations with rigid horizontal boundaries [3,4]. On the contrary, the correct performed simulations with rigid boundary conditions on the horizontal plates have a good agreement with experimental data [5,6,8]. On the other hand, it seems natural that results of simulations with free and rigid boundary conditions must draw together at enough high supercriticality values. The question of togetherness of solutions with free and rigid boundary conditions is practical, since using of free boundary conditions very simplifies the DNS of turbulent convection, simple and efficient numerical algorithms are generated using the formulas of linear stability theory [9]. The using of formulas from linear stability theory guarantees the exact conformity of spectral characteristics of differential problem and numerical method. The work [9] contains the some results of comparative analysis of this spectral method and finite-difference used for simulation of turbulent convection [10].

The aim of this work is linear and nonlinear (on the model nonlinear system of equations) analysis of spectral numerical method suggested in work [9] for simulation of convection with free horizontal boundaries, performing of numerical calculations of turbulent convection at supercriticality of order of 1000 times critical value, comparing of derived results with experimental data and numerical results of other authors.

Problem formulation and numerical method

Turbulent convective flow in a horizontal layer numerically is simulated at heating from below. The fluid is viscous and incompressible. The flow is time-dependent and two-dimensional. Boundaries of a layer are isothermal and free from shearing stresses. The model Boussinesq is used without semiempirical relationships.

The dimensionless input set of equations given in terms of deviations from an equilibrium solution is of the form [9]:

$$\omega_t + \frac{1}{\Pr}(\varphi_y \omega_x - \varphi_x \omega_y) = \Delta\omega + RaQ_x,$$
$$\Delta\varphi = -\omega, \tag{1}$$
$$Q_t + \frac{1}{\Pr}(\varphi_y Q_x - \varphi_x Q_y) = \frac{1}{\Pr}\Delta Q - \frac{1}{\Pr}\varphi_x,$$

where φ is a stream function, ω is the vortex, Q is the temperature deviation from equilibrium profile (the total temperature being T = 1 - y + Q), $\Delta f = f_{xx} + f_{yy}$ is the Laplace operator, $Ra = g\beta H^3 dQ/\chi\nu$ is the Rayleigh number, $\Pr = \nu/\chi$ is the Prandtl number, g is the gravitational acceleration, β, ν, χ are the coefficients of thermal expansion, kinematics viscosity and thermal conductivity, respectively, H is the layer thickness and dQ is the temperature difference on the horizontal boundaries.

The required values ω, φ and Q are to be sought in the form:

$$\omega(t,x,y) = \sum_{k,m} \omega_{km}(t)\rho_k \cos(\alpha kx)\sin(\pi my),$$
$$\varphi(t,x,y) = \sum_{k,m} \frac{\omega_{km}}{S_{km}}\rho_k \cos(\alpha kx)\sin(\pi my),$$
$$Q(t,x,y) = \sum_{k,m} Q_{km}(t)\sin(\alpha kx)\sin(\pi my),$$

where $\alpha = \pi/L$ is the wave number, and $\rho_k = \{0.5$ (at k = 0, N) and 1 (at $1 \le k \le N-1$)$\}$, $0 \le k \le N$, $1 \le m \le M-1$, $S_{km} = \alpha^2 k^2 + \pi^2 m^2$. Solution is periodic, but we consider this solution only in half of period in X – direction, therefore the periodic problem changes on the problem with boundary conditions on the side walls, according to form of solution.

Following a general ideology of the splitting method, transition from layer n to layer n+1 on the time is performed in two steps. On the first step, we take into account a linear development of perturbations without interaction between harmonics.

<u>*Step 1.*</u>

$$\omega_t = \frac{1}{2}\Delta\omega + RaQ_x,$$
$$\Delta\varphi = -\omega,$$
$$Q_t = \frac{1}{2\Pr}\Delta Q - \frac{1}{\Pr}\varphi_x.$$

This linear system decides on the analytic formulas in spectral space.

The second step takes into account the nonlinear convection transfer, i.e., the interaction between harmonics.

<u>*Step 2.*</u>

$$\omega_t + \frac{1}{\Pr}(\varphi_y \omega_x - \varphi_x \omega_y) = \frac{1}{2}\Delta\omega,$$
$$\tag{2}$$
$$Q_t + \frac{1}{\Pr}(\varphi_y Q_x - \varphi_x Q_y) = \frac{1}{2\Pr}\Delta Q.$$

Here we use a finite-difference scheme of alternating directions for solving the system equations of nonlinear convective transfer in physical space. This scheme was used for simulation of turbulent convection [10]. For transition from spectral space into physical space and back, standard programs of FFT were used. The numerical method has the first order of approximation on the time and the second order of approximation on the space variables.

The coefficients φ_x and φ_y in (2) defined: 1. By value of the stream function with layer n on the time (Scheme 1). 2. By value of the stream function after first step of splitting (Scheme 2). 3. By value of arithmetic mean of stream function on the layers n and n+1 on the time (Scheme 3). Realization of Scheme 3 demands the introduction of iteration process.

Linear analysis

Linear analogs of differential system (1) and numerical method are considering and calculating them spectral characteristics. By closeness them we may estimate the accuracy of reproduction of infinitesimal disturbances by numerical method.

We consider the solutions of linear problems in a view:

$$\omega(t,x,y) = a e^{-\lambda t} \cos(\alpha x) \sin(m \pi y),$$
$$Q(t,x,y) = b e^{-\lambda t} \sin(\alpha x) \sin(m \pi y),$$
$$\varphi(t,x,y) = \frac{\omega(t,x,y)}{\alpha^2 + m^2 \pi^2},$$

here a and b are constants and λ is eigenvalue.

The fig.1 represents the spectral curves for differential problem (solid line), suggested spectral method (sign ●) and finite-difference method (dash line)[10] for the first three modes (m = 1,2,3) $\lambda = \lambda(\alpha)$ (for r = 1000, Pr = 1, N = 64, M = 16, $\tau = 4\ 10^{-4}$). On the fig. 2 most unstable mode is showed (m = 1) for various spectral methods, here solid line represents the differential problem, sign ● - present method, dash line - Orszag method (changed for 2-D)[2], dadot – [11], sign ○ - [7].

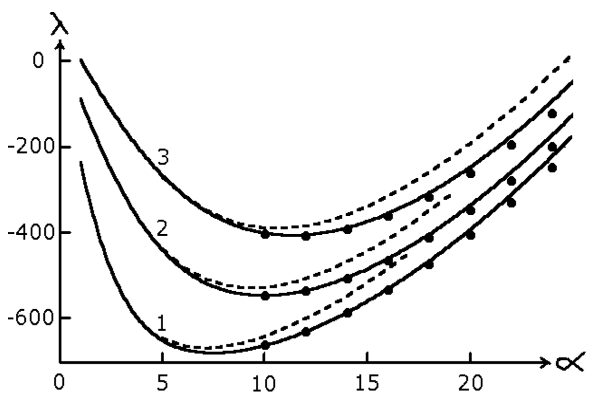

Fig.1. Spectral curves.

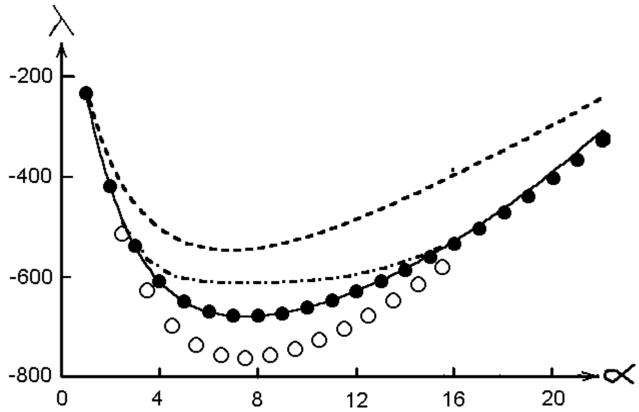

Fig.2. Spectral curves for the spectral methods.

The fig. 1 shows that suggested spectral method has more accuracy than finite-difference. The fig.2 shows that suggested spectral method exact reproduces the spectral characteristics of differential problem even at big step on the time. It is the consequence of using of analytic formulas on the first step of the splitting. The exact reproduction of spectral characteristics guarantees the exact reproduction of infinitesimal disturbances.

Unfortunately, the linear analysis do not allows investigating the approximation of nonlinear terms on the time (Scheme 1 – 3).

Nonlinear analysis

We perform the nonlinear analysis on the model nonlinear system of equations:

$$\omega_t + |\varphi_y|\omega_x + |\varphi_x|\omega_y = \Delta\omega + RaQ_x,$$
$$\Delta\varphi = -\omega, \qquad (3)$$
$$Q_t + |\varphi_y|Q_x + |\varphi_x|Q_y = \Delta Q - \varphi_x,$$

The system (3) has private solution in waveform:

$$\omega(t,x,y) = \rho(t)e^{i(\psi(t)+\alpha x+\beta y)},$$
$$Q(t,x,y) = -i\eta(t)e^{i(\psi(t)+\alpha x+\beta y)},$$
$$\varphi(t,x,y) = \frac{\omega(t,x,y)}{\alpha^2+\beta^2}.$$

$$\rho(t) = \frac{1}{2}\{C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}\}, \eta(t) = \frac{1}{2\sqrt{SRa}}\{C_1 e^{-\lambda_1 t} - C_2 e^{-\lambda_2 t}\},$$

$$\psi(t) = \psi_0 + A\{\frac{C_1}{\lambda_1}e^{-\lambda_1 t} + \frac{C_2}{\lambda_2}e^{-\lambda_2 t}\}, \lambda_i = S + (-1)^i\alpha\sqrt{Ra/S}, i = 1,2.$$

here $A = 2\alpha\beta/S$, $S = \alpha^2+\beta^2$, C_1 and C_2 are arbitrary constants. The same solutions have and suggested spectral method, then these solutions may compare. In a similar way, we analyzed finite difference schemes for nonlinear equation with oscillating viscosity [12] and finite difference method for calculation of viscoelastic flows [13].

Let the values of ρ and η on the layer n on the time are well known, we will derive the expressions in form of power series for values of ρ and η on the layer n+1 on the time.

After cumbersome calculations with Maple V program we have:

$$\rho_d^{n+1} - \rho_{sp}^{n+1} = O(\tau^3), \ \eta_d^{n+1} - \eta_{sp}^{n+1} = O(\tau^3), \ \psi_d^{n+1} - \psi_{sp}^{n+1} = O(\tau^k),$$

here k = 2 for Schemes 1 and 2 and k = 3 for scheme 3.

These formulas show that amplitudes ρ and η calculate with the same accuracy by Schemes 1-3, and that using of Schemes 1 and 2 leads to decrease of calculation accuracy only in phase speed of solution. For direct numerical simulation (DNS) of turbulent convection using of Schemes 1 and 2 is expedient.

DNS of turbulent convection

We have simulated the Raylegh-Benard convection with α = 1, N = 64, M = 16, Pr = 2 or 10 and r = Ra/Racr up to 1000 times critical value. The simulations were finished if increasing of time interval do not leaded to changing of mean Nusselt number.

The calculated Nusselt numbers at $5 \le r \le 50$ (Pr = 10) have a good agreement with data of works [3,11,14].

The fig.3 represents isotherms of full temperature, above – experimental interferogram at r = 2.2 in water [15], below – stationary solution derived by suggested method at r = 2 and Pr = 2.

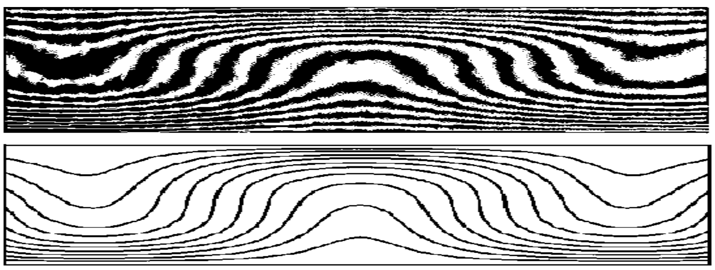

Fig.3. Experimental and calculated interferograms of full temperature.

The fig. 3 shows the startling visual coincidence of experimental and calculated temperature isotherms.

The fig. 4 represents profile of full temperature, here sign ● - results of present work at r = 1000 and Pr = 10, sign ○ - experimental results [16] at r = 1500 in air, solid line – experimental results [17] at r =1000 in water. The fig.5 shows profile of r.m.s. temperature fluctuations, here sign ● - results of present work at r = 400 and Pr = 10, solid line – numerical results [8] at r = 370 in air, sign ◊ - experimental results [16] at r = 370 in air, dash line – experiment [18] at r = 400 in air.

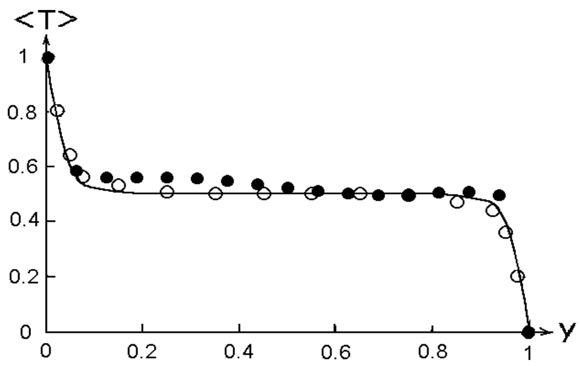

Fig.4. Profile of full temperature.

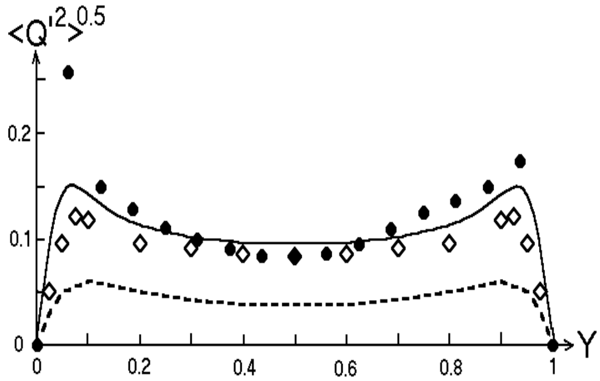

Fig.5. Profile of r.m.s. temperature fluctuations.

The fig.4 shows that result of present work corresponds to experimental results in water and air at supercriticality r ≈ 1000. The fig.5 shows that derived results correspond to numerical and experimental results of other authors at r ≈ 400. The big deviations may be seen only in single point near bottom boundary at y ≈ 0. The fig.5 shows also a big scatter in experimental data.

The table 1 represents the Nusselt numbers at r = 1000 derived by suggested spectral method (Pr = 10), data of numerical simulation with free boundary conditions [3], data of numerical simulation with rigid boundary conditions [6] and data of experimental works in water [17,19].

Table 1. Comparison of Nusselt number at r = 1000 in water.

Present	[3], free	[6], rigid	[17], exp.	[19], exp.
9.808	22.401	9.365	9.887	9.705

The arithmetic mean of experimental values from table 1 is equal to 9.796, it differs weakly from result of present work. The value of Nusselt number from work [3] is significantly overstated.

Conclusion

The suggested spectral method reproduces exactly the spectral characteristics of differential problem even at big step on the time. It guarantees exact representation of development of infinitesimal disturbances of equilibrium (trivial for system (1)) solution.

Nonlinear analysis of suggested numerical method shows that calculation of coefficients of nonlinear transfer (second step of splitting (2)) by the values with n time layer (Scheme 1) and by the values after first step of splitting (Scheme 2) leads to decrease of calculation accuracy only of phase speed of solution. For direct numerical simulation of turbulent convection using of Schemes 1 and 2 is expedient.

Results of calculations correspond to experimental data in turbulent convection and to results of numerical investigations of other authors.

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