A QUASI-SPHERICAL COORDINATE SYSTEM AND ITS APPLICATION TO THE DETERMINATION OF VERTEX-TYPE SINGULARITIES

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Summary To avoid the inherent singularity of the spherical coordinates at their poles, a new quasi-spherical coordinate system is developed. In this coordinate system, a finite element procedure is proposed to determining the eigensolutions at three-dimensional vertices in which the field variables are proportional to the (λ+1)-th power of the distance from the vertices. The resulting global equation is a second order characteristic matrix equation. Several demonstrating problems are investigated. It can be seen that the formulation yield satisfactory results.

1. INTRODUCTION

Cracks, notches and multi-material joints are often found in modern engineering structures and materials. Thorough understanding of the singular stress fields around them is needed for the reliability prediction of these structures and materials during their service life. Many studies about singular asymptotic stress fields in linear elastic materials have been carried out in recent two decades. Most of the researches in this area have focused on two-dimensional geometry. Attention has also been given to three-dimensional conical configurations. The present concern is vertex-type singularity of the stress in the form of:

$$\rho^{\lambda}\mathbf{S}(\phi_z,\theta_z) \tag{1}$$

where (ρ,ϕ_z,θ_z) are the standard spherical coordinates with their origin at the vertex of the conical material domains.

Owing to the intrinsic mathematical complexity, vertex-type singularities were mostly identified by numerical methods [1]—[6]. These numerical methods are all developed in the standard ρ-ϕ_z-θ_z spherical coordinates and the concerned displacement is the one defined with respect to ρ, ϕ_z and θ_z. Moreover, the surface of the spherical domain is meshed uniformly in the ϕ_z-θ_z plane. Hence, the mesh in the physical space is densest and coarsest at the vicinity of the poles ($\phi_z = 0$ or π) and at the equator ($\phi_z = \pi/2$) respectively. All nodes at $\phi_z = 0$ (or $\phi_z = \pi$) in the ϕ_z-θ_z plane represent a single physical point which is the pole on the sphere. Interestingly, different values of displacement are yielded at different nodes at $\phi_z = 0$ (or $\phi_z = \pi$). This violates our physical awareness. Moreover, some strain components and thus the integrands in the weak form contain the $1/\sin\phi$ terms which are unbounded at the poles. Accurate numerical integration is difficult. Slow convergence was reported in Bazant [1] and Bazant & Estenssoro [2] by using finite difference and finite element methods. To yield accurate solutions, extrapolation was employed. Meanwhile, Pageau & Biggers [6] avoided the problem by excluding the small cones of material at the poles in their discretization. Nevertheless, their results are rather sensitive to the order of integration. The observation is probably due to the high strain energy density gradient at the vicinity of the poles due to the $1/\sin\phi$ terms. On the other hand, Ghahremani & Shih [4] overcame the problem by directly dividing the spherical surface into quadratic triangular element to reduce the nonuniformity arising from using uniform meshes in the ϕ_z-θ_z plane. To overcome the singularity at the poles, local spherical coordinates for each of the elements are defined in such a way that the element centre is close to the point wherein the local ϕ_z and θ_z are equal to $\pi/2$. Unfortunately, compatibility is not satisfied in the method.

In present study, in order to avoid the inherent singularity of the spherical coordinates at the poles, a three quasi-spherical coordinates system is employed to describe the three-dimensional space. A new finite element formulation for solving vertex-type singularities is developed in the frame of this new coordinates system. Moreover, the nodal d.o.f.s are the generalized displacement components along the global Cartesian coordinate axes. It will be seen that the formulation is compatible and insensitive to the order of numerical integration. Problems on quarter-infinite cracks as well as 1/8-1/8, 1/8-4/8 and 1/8-2/8 bi-material junctions are studied. The computed results are in closed agreement with those reported in the literature.

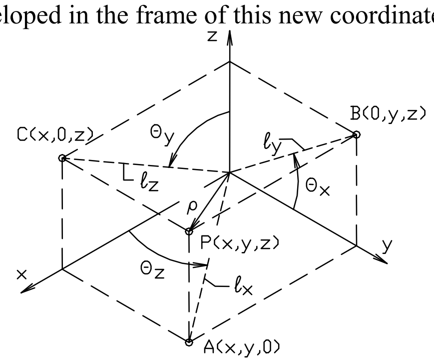

Fig. 1

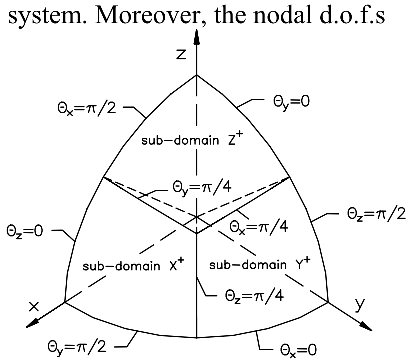

Fig. 2

2. A QUASI-SPHERICAL COORDINATE SYSTEM

In order to secure the mesh compatibility and the result insensitivity to the order of numerical integration, a complementary quasi-spherical coordinate system is here proposed. Besides the angular coordinate θ_z used in the standard spherical coordinates, θ_x and θ_y are introduced in this complementary quasi-spherical coordinate system (see Fig. 1). The three-dimensional space is divided into the following six sub-domains:

Domain $X^+: 0 \le \rho \le \infty, \pi/4 \le \theta_y \le 3\pi/4, -\pi/4 \le \theta_z \le \pi/4$; Domain $X^-: 0 \le \rho \le \infty, -3\pi/4 \le \theta_y \le -\pi/4, 3\pi/4 \le \theta_z \le 5\pi/4$;

Domain $Y^+: 0 \le \rho \le \infty, \pi/4 \le \theta_z \le 3\pi/4, -\pi/4 \le \theta_x \le \pi/4$; Domain $Y^-: 0 \le \rho \le \infty, -3\pi/4 \le \theta_z \le -\pi/4, 3\pi/4 \le \theta_x \le 5\pi/4$;

Domain $Z^+: 0 \le \rho \le \infty, \pi/4 \le \theta_x \le 3\pi/4, -\pi/4 \le \theta_y \le \pi/4$; Domain $Z^-: 0 \le \rho \le \infty, -3\pi/4 \le \theta_x \le -\pi/4, 3\pi/4 \le \theta_y \le 5\pi/4$. (2)

Domain X's, Y's and Z's are described by the quasi-spherical coordinates (ρ, θ_y, θ_z), (ρ, θ_z, θ_x) and (ρ, θ_x, θ_y), respectively. Fig. 2 illustrates how a typical octant is partitioned.

3. FINITE ELEMENT FORMULATION IN THE QUASI-SPHERICAL COORDINATE SYSTEM

The weak form of the eigen problems with vertex-type singularities is ([1], [2]):

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_S (\delta\mathbf{u}^T \mathbf{t})\Big|_{\rho=R} dS = 0 \tag{3}$$

in which δ is the virtual symbol, V denotes volume, S denotes the spherical surface at radius $\rho = R$, $\boldsymbol{\varepsilon}$ is the vector of strain components, $\boldsymbol{\sigma}$ is the vector of stress components, $\mathbf{u}$ is the displacement and t is the surface traction. Strain, stress and surface traction are all derived from the displacement. It should be noted that the above weak form is different from the virtual work statement wherein the surface traction is prescribed.

The displacement is assumed to be:

$$\mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \mathbf{u}_o + \rho^{\lambda+1} \sum_{i=1}^{m} N_i \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}_i \tag{4}$$

where the nodal d.o.f.s are the generalized Cartesian displacement components U, V and W which are interpolated inside each element to give the Cartesian displacement components u, v and w; $\mathbf{u}_o$ is the displacement at $\rho = 0$; N's are the interpolation functions defined in the new quasi-spherical coordinate system.

Following the standard procedures that used to formulate an assumed displacement isoparametric finite element, a second order characteristic matrix equation is obtained. It can be transformed into the standard first order characteristic matrix equation by introducing a dummy vector and the later can be solved by standard numerical subroutines.

5. CONCLUSIONS

With the asymptotic assumption of the displacement that leads to the vertex-type stress singularity, a two-dimensional finite element formulation is derived. Unlike the previously proposed formulations which suffer from either mesh incompatibility or the inherent singularity of the standard spherical coordinate system at its poles, the proposed formulation employs three quasi-spherical coordinate systems to describe different regions of the three-dimensional space. Moreover, the unknowns are the generalized displacement components along the Cartesian coordinates instead of the ones defined with respect to the standard or quasi- spherical coordinates. In this light, the shortcomings of the previous formulation are circumvented. The subsequent characteristic matrix equation is quadratic and can be transformed into a linear one by introducing a dummy vector of unknowns. Benchmark problems including quarter infinite crack in isotropic material (Fig.3), quarter infinite crack in orthotropic material, tri-material free edge and solutions, if any.

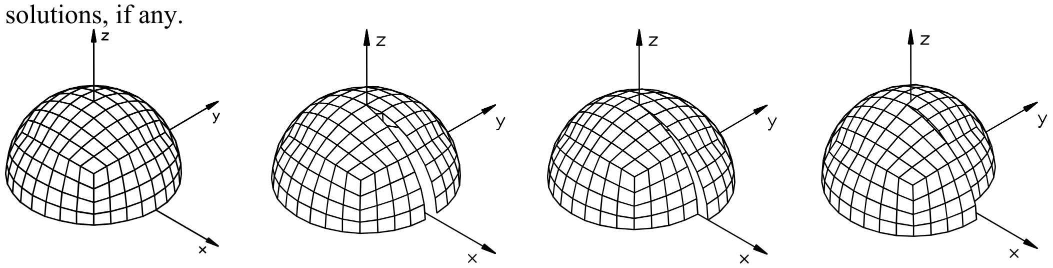

Fig. 3

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