Estimation of Principal Axes of Inertia

Masashi Sato* , Yasusuke Jinnouchi*

* *Kitakyushu College of Technology, 5-20-1 Shii, Kokuraminami, Kitakyushu, JAPAN*

Summary This paper presents a new experimental method to measure the principal axes of a rigid body without the principal axis transformation. A rigid body is suspended from a vertical axle by a slender rod. The axle rotates with a constant angular velocity. The experimental results for a cylinder and a rectangular prism agreed well with the theoretical results. And the principal axes of a golf club head were estimated experimentally.

INTRODUCTION

In the analysis of motion of a rigid body, it is necessary to know the inertia characteristics, that is, the moments and the products of inertia, the principal axes, the principal moments of inertia as well as the mass and the mass center of the rigid body. Although the estimation of the mass and the mass center is not difficult in general, the experimental measurement of the moments of inertia involves difficulties [1, 2].
Incidentally, it is well known that a rigid body under no moment can rotate about a fixed centroidal axis, if and only if that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid and unstable if it coincides with the intermediate axis [3]. We took note of this fact, and tried to estimate the principal axes of a rigid body without the principal axis transformation. Our goal is to propose a new experimental method to estimate the principal axes.

MOTION OF A SUSPENDED RIGID BODY SYSTEM

Motion of a rigid body

Figure 1 shows a system considered. A rigid body suspended by a slender rod rotates at a constant angular velocity ω about the z-axis fixed to space. $0-\xi\eta\zeta$ is the moving coordinate fixed to the rigid body, and 0 is the mass center of the body. the rod is attached to the body at point B on η-axis. Point A denotes the intersection of η-axis and z-axis. The angle α between η-axis and horizontal plane and the inclination θ are assumed to be constant. Air drag and friction are neglected.The equation of motion of the rigid body shown in Figure 1 is

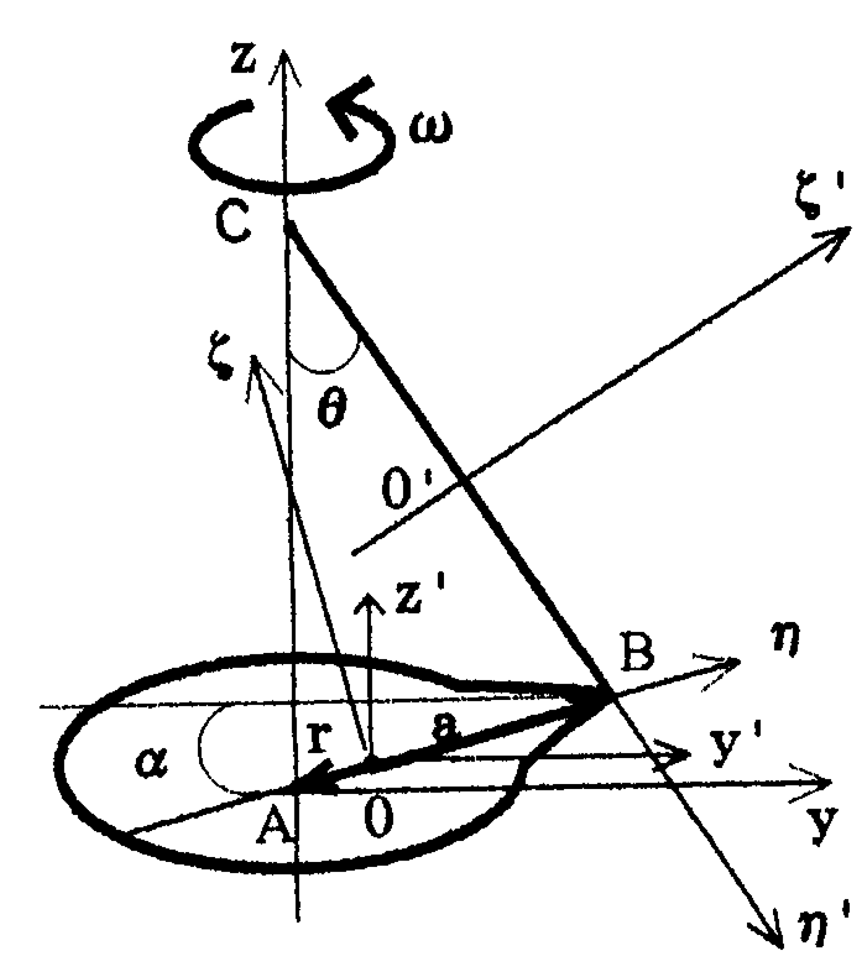

Fig.1 System considered

$$\frac{d^*\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} = \boldsymbol{N} \tag{1}$$

where $\boldsymbol{L}$, $\boldsymbol{\omega}$, and $\boldsymbol{N}$ are the angular momentum, the angular velocity and the external moment respectively. They are presented in the form referred to the moving coordinate $0-\xi\eta\zeta$ as follows :

$$\boldsymbol{L} = (A_{\xi\eta}\,\omega\sin\alpha + A_{\zeta\xi}\,\omega\cos\alpha)\boldsymbol{e}_\xi + (A_{\eta\eta}\,\omega\sin\alpha + A_{\eta\zeta}\,\omega\cos\alpha)\boldsymbol{e}_\eta + (A_{\eta\zeta}\,\omega\sin\alpha + A_{\zeta\zeta}\,\omega\cos\alpha)\boldsymbol{e}_\zeta$$

$$\boldsymbol{\omega} = \omega\sin\alpha\,\boldsymbol{e}_\eta + \omega\cos\alpha\,\boldsymbol{e}_\zeta$$

$$\boldsymbol{N} = ma(g + r\omega^2\sin\alpha)\cos\alpha\,\boldsymbol{e}_\xi$$

where A_{ij} is the component of the inertia tensor, $\boldsymbol{e}_i$ is the unit vector of i-axis and g is the acceleration of gravity. In the present case α, θ and ω are constant. Hence the first term on the left hand side of Eq.(1) vanishes. Substituting $\boldsymbol{L}$, $\boldsymbol{\omega}$ and $\boldsymbol{N}$ into Eq.(1), the following three equations are obtained.

$$\omega^2\left\{\frac{1}{2}(A_{\zeta\zeta} - A_{\eta\eta})\sin 2\alpha - A_{\eta\zeta}\cos 2\alpha\right\} = mga\cos\alpha + mra\omega^2\sin\alpha\cos\alpha \tag{2}$$

$$\omega^2\cos\alpha\{A_{\xi\eta}\sin\alpha + A_{\zeta\xi}\cos\alpha\} = 0 \tag{3}$$

$$-\omega^2\sin\alpha\{A_{\xi\eta}\sin\alpha + A_{\zeta\xi}\cos\alpha\} = 0 \tag{4}$$

Then the products of inertia $A_{z'x'}$ and $A_{y'z'}$ are derived as follows.

$$A_{z'x'} = -\int z'x'dm = -\int(\eta\sin\alpha + \zeta\cos\alpha)\xi dm = A_{\xi\eta}\sin\alpha + A_{\zeta\xi}\cos\alpha \tag{5}$$

$$A_{y'z'} = -\int y'z'dm = A_{\eta\zeta}\cos 2\alpha - \frac{1}{2}(A_{\zeta\zeta} - A_{\eta\eta})\sin 2\alpha = -ma\cos\alpha(\frac{g}{\omega^2} + r\sin\alpha) \quad (6)$$

If the mass of the slender rod connected to the rigid body is negligible,

$$r = \frac{ag}{l\omega^2\cos\theta - g} \quad (7)$$

Finally, we may observe that $A_{z'x'}$ is equal to zero from Eq.(3) and that $A_{y'z'}$ also approaches zero if ω is sufficiently large. Then the z' axis becomes one of the principal axes and coincides with the z axis. In the successive experiment, if another stable rotation can be made, the principal axes will be obtained.

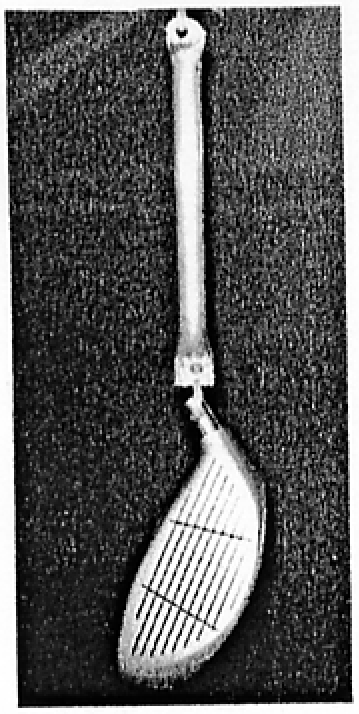

Fig.2 Photograph of the suspended rigid body

Motion of a slender rod BC

As the centroidal frame $0' - \xi'\eta'\zeta'$ of the slender rod BC (Fig.1) coincides the principal axes of inertia, the principal moments may be expressed as $I_{\xi'\xi'} = m'l^2/12$, $I_{\eta'\eta'} = 0$, $I_{\zeta'\zeta'} = m'l^2/12$, where m' is the mass and the l is the length of the rod. Substituting $\omega \times \boldsymbol{L}$ and $\boldsymbol{N}$ of the rod into Eq.(1), the equation of motion in ξ' direction may be written in the form

$$\left(\frac{r}{r+a} + \frac{1}{3}\frac{m'}{m}\right) = \frac{g}{l\omega^2\cos\theta}\left(1 + \frac{1}{2}\frac{m'}{m}\right) \quad (8)$$

In case the magnitude of m' cannot be disregarded, it is found that r does not become zero, even if ω becomes very large, and Eq.(8) reduces to

$$r = -\frac{1}{3}\frac{m'}{m}a \Big/ \left(1 + \frac{1}{3}\frac{m'}{m}\right) \quad (9)$$

In order that $A_{y'z'}$ may become zero, substituting Eq.(9) into Eq.(6), ω is obtained as

$$\omega^2 = \frac{3g}{2\mu}\left(\frac{1+\frac{\mu}{3}}{a\sin\alpha} + \frac{1+\frac{\mu}{2}}{l\cos\theta}\right)\left(1 \pm \sqrt{H}\right) \quad (10)$$

where

$$H = 1 - \frac{\frac{4\mu}{3}\left(1+\frac{\mu}{2}\right)(l\cos\theta)(a\sin\alpha)}{\left\{l\cos\theta\left(1+\frac{\mu}{3}\right) + a\sin\alpha\left(1+\frac{\mu}{2}\right)\right\}^2} \ , \ \mu = \frac{m'}{m} \quad (11)$$

Fig.3 Photograph of the rotating rigid body

If the magnitude of m' may be disregarded, Eq.(8) is led to Eq.(7).

EXPERIMENTAL RESULTS AND CONCLUSIONS

Figure 2 shows that the suspended rigid body is at rest and Figure 3 shows it is rotating. The results indicate that this method is useful to estimate the principal axes.

References

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