TRUNCATED ELASTIC WEDGE UNDER TORSIONAL LOAD

Vyacheslav V. Lyakh
Department of Theoretical & Applied Mechanics, Kiev National Taras Shevchenko University, 01033 Kiev, Ukraine

Summary This paper deals with an analytical solution of the boundary-value problem of plane elasticity for a truncated infinite wedge of an arbitrary opening angle 2α. The flanks of the wedge are free of traction. Its circular boundary is subjected to torsional load due to the given tangential displacements or the moment-replacement loading prescribed. The main goal of the paper is to verify whether the Carothers paradox is actual when the statement of the Carothers problem is modified and more rigorous. Two powerful methods, viz. the method of superposition and the method of homogeneous solutions, are introduced and compared. By means of them the boundary-value problem amounts to solving an infinite integro-algebraic system of equations and an infinite system of algebraic equations, respectively. Our numerical simulations with these systems provide graphical results. The distributions of stresses in some principal cases are presented. Numerical results turn out to be in a complete agreement with results by Neuber [1].

SHORT REFERENCES AND STATEMENT OF THE PROBLEM

A plane elastostatic problem for an elastic wedge loaded by a concentrated moment at its apex provides an example of violation of the Saint-Venant principle when the apex angle 2α of the wedge is over than the half-plane. This circumstance is due to Sternberg and Koiter [2]. They examined the non-degenerate modified problem, in which the couple is replaced by a statically equivalent continuous load distributed on the flanks close by the apex. As distinct from Sternberg and Koiter, Neuber [1] considered the problem for a truncated wedge. He showed the method of construction of an applicable solution for any apex angles in the range $\pi \leq 2\alpha \leq 2\pi$ despite the failure of the Saint-Venant principle. By all means further advancement in explanation of the Carothers paradox is possible provided that the wedge problem is modified in a certain manner and rigorously formulated (see Markenscoff [3]).

Therefore, this paper addresses two modified wedge problems. In the both cases we consider the truncated elastic wedge occupying the domain $\varepsilon \leq r < \infty$, $-\alpha \leq \vartheta \leq \alpha$ in the polar coordinates system (r, ϑ) with free surfaces $\vartheta = \pm\alpha$ (see Figure 1 and conditions (1)).

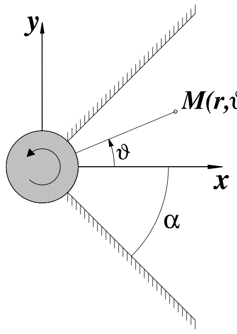

Figure 1. Truncated elastic wedge.

$$\sigma_\vartheta = 0, \quad \tau_{r\vartheta} = 0, \qquad \varepsilon \leq r < \infty, \quad \vartheta = \pm\alpha. \tag{1}$$

The difference lies in the boundary conditions on the circular surface.

Bonded contact with a rigid shaft (mixed problem)

Let us suppose that the truncated elastic wedge is bonded to a rigid shaft (cylinder) of radius ε (see Figure 1). The shaft is rigidly joined with the wedge across the contact zone $r = \varepsilon$, $-\alpha \leq \vartheta \leq \alpha$. A torsional moment M_0 is acting on the shaft which as consequence has rotated through a certain angle φ_0 counterclockwise. Therefore, the displacements in the contact zone are prescribed as

$$u_\vartheta = \varepsilon \cdot \varphi_0, \quad u_r = 0, \qquad r = \varepsilon, \quad -\alpha \leq \vartheta \leq \alpha, \tag{2}$$

As usually, the displacements are assumed vanishing at infinity $r \to \infty$.

This problem is natural in its technical aspect, and it agrees with the approach by Neuber [1]. The solution of the problem will be a function of the angle φ_0, and there is an one-to-one correspondence between φ_0 and the torsional moment M_0. Therefore, full qualitative agreement with Neuber [1] is expected.

On the other hand, if the Carothers paradox is actual then for the given φ_0 the moment M_0 transferred from the shaft to the truncated wedge is not a monotonous increasing function of the angular coordinate α. This matter is found out when solving the problem.

Moment-replacement loading

In contrast to Sternberg and Koiter [2], the replacement loading is applied on the surface $r = \varepsilon$. The normal σ_r and tangential $\tau_{r\vartheta}$ stresses are specified so that the resultant force of external tractions vanishes. Specifically, they are given as $\sigma_r = -X_n \cdot \cos\vartheta$, $\tau_{r\vartheta} = X_n \cdot \sin\vartheta$, where X_n is a given odd function. By replacing X_n with its Fourier series, the boundary conditions at $r = \varepsilon$ are written as

$$\sigma_r = \cos\vartheta \sum_{m=1}^{\infty} a_m \sin\Big(\frac{m\pi}{\alpha}\,\vartheta\Big), \quad \tau_{r\vartheta} = -\sin\vartheta \sum_{m=1}^{\infty} a_m \sin\Big(\frac{m\pi}{\alpha}\,\vartheta\Big), \qquad r = \varepsilon, \quad -\alpha \le \vartheta \le \alpha. \tag{3}$$

METHOD OF SOLUTION

The method of homogeneous solutions (see [4]) and the method of superposition (see [5]) are employed. They, respectively, allow to transform the boundary-value problem to an infinite system of algebraic equations and to an infinite integro-algebraic system of equations.

It is more convenient to apply the method of homogeneous solutions if the opening angle of the wedge 2α equals π or 2π. The last is important particular case when the truncated wedge degenerates into an elastic plane with a semi-infinite slit (see [6]). If 2α differs from π or 2π roots of characteristic equations are complex. In this connection natural difficulties were overcome.

RESULTS

Proceeding from the method of homogeneous solutions the mixed problem has been reduced to solving a singular infinite system of linear algebraic equations that can be brought to a regular form. An asymptotic analysis of singular system's equations and a regular system's solution has been carried out. On the basis of a numerical solution of the regular infinite system normal σ_r and shearing $\tau_{r\vartheta}$ stresses are plotted as a function of the angular coordinate ϑ on a joint area between the elastic wedge and the rigid shaft. Finally, the graphs display somewhat unexpected results: these relations are found in a complete qualitative agreement with Neuber's deductions (see Figure 2).

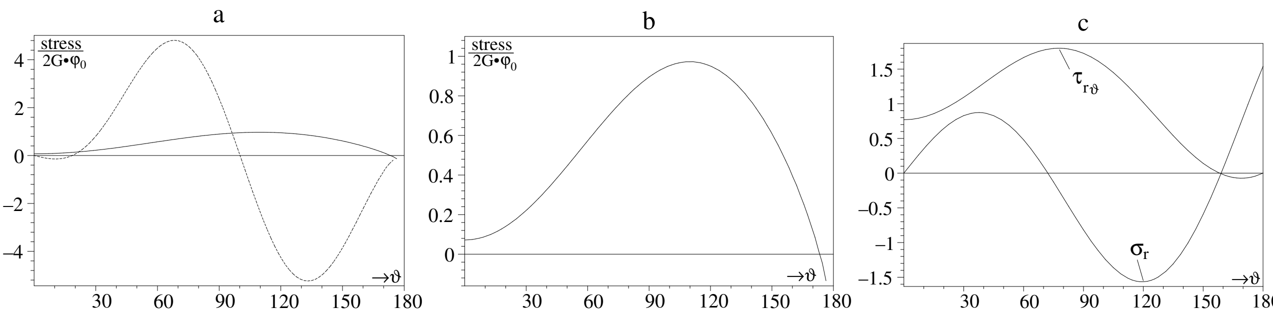

Figure 2. Stress distribution along the force transfer surface calculated for $\alpha = \pi$: (a) dashed and solid lines represent normal and shearing stresses respectively; (b) shearing stress scaled down; (c) according to Neuber [1].

As in a number of other problems, including bonded contact between elastic bodies and rigid boundaries, there is a certain inconsistency in the shear-stress behavior at the location $r = \varepsilon$, $\vartheta = \pm\alpha$. This character of the stress field is inevitable. Therefore, the points $r = \varepsilon$, $\vartheta = \pm\alpha$ are treated as singular, in which characteristics of the stress field are uncertain and fundamentally different in their close proximity.

References

[1] Neuber H.: Lösung des Carothers - Problems mittels Prinzipien der Kraftübertragung (Keil mit Moment an der Spitze). *Z. angew. Math. Mech.* 43 (1963) 211-228.

[2] Sternberg E., Koiter W. T.: The wedge under a concentrated couple: a paradox in the two-dimensional theory of elasticity. *Trans. ASME. J. Appl. Mech.* 25 (1958) 575-581.

[3] Markenscoff X.: Some remarks on the wedge paradox and Saint-Venant's principle. *Trans. ASME. J. Appl. Mech.* 61 (1994) 519-523.

[4] Lourje A. I., *Theory of Elasticity.* Moscow: Nauka (1970) 939pp. (in Russian).

[5] Grincenko V. T., *Equilibrium and Stady-State Oscillations of Elastic Bodies of Finite Dimensions.* Kiev: Naukova Dumka (1978) 264pp. (in Russian).

[6] Ulitko A. F., Lyakh V. V.:Torsional load transfer from a rigid shaft to an elastic plane with a slit. *J. Engng Math.* 46 (2003) 395-408.