AXISYMMETRIC PROBLEM FOR AN ELASTIC MEDIUM WITH A SPHERICAL INCLUSION WHEN THERE IS A CRACK AT THE INTERFACE

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Summary A problem on the stressed state of an elastic medium with a spherical inclusion with a crack at the interface boundary is solved by exact methods of the linear theory of elasticity. At first the problem is reduced to an interrelated system of coupled integral equations with respect to the Legendre functions, and then to a system of singular integral equations with respect to two unknown functions. The behaviour of the equation solutions is studied near the interface circle of a spherical section. The case is examined when the cross-section surfaces are under normal internal pressure of constant intensity.

INTRODUCTION

As is well known in most cases the catastrophic destruction of constructions is caused by some hidden internal defects. The internal cracks, in the form of material solidity breaks have been examined by scientific literature for quite a long time. The problems on stress-strained state of elastic finite and infinite bodies with flat cracks of disk-shaped or elliptic forms, are among the best researched ones. More than a hundred publications deal with this issue. However, according to experimental analysis of the surfaces of damaged parts, the initial surfaces of the material breaks were of spherical or ellipsoidal shape, that is they were volumetrical, not flat. To evaluate the strength of material with internal cracks, one can start with the solution of a class of problems within the elasticity theory for three-dimensional bodies weakened by spatially-bent cracks. Such cracks could be modelled by cuts on a part of some surface of revolution with its non-zero curvature. In this case, there is a possibility to vary geometrical parameters of the surface and, by doing this, to bring them closer to the geometry of real cracks. The experimental study of composite materials emphasise the practical significance of this class of problems. Specifically, the heterogeneous media are being filled with the particles of spherical or ellipsoidal form (for example, wolfram-carbide matrix reinforced by diamond grains). Their mechanical characteristics depend, to a considerable degree, on the material solidity break, which appears, as a rule, on the interphase boundary and is located on the parts of the spherical or ellipsoidal surface. The theory of spatial cracks presents quite natural tendencies, when complicated geometry of cuts prompts the need to develop more complex mathematical methods and to increase significantly the number of mathematical operations needed for their analysis. This, obviously, provides an explanation to the fact that just a few publications deal with the studies of stress fields and the displacements of elastic bodies with volumetrical cuts, although the importance of such studies was repeatedly emphasised in research literature.The present paper examines the axisymmetric elastostatic problem for two-compound body with a crack on a part of spherical surface located on the boundary of the division of elastic characteristics of materials. This problem is linked to the study of stressed state of high-strength composite materials with low-percentage content of the spherical dispersed particles.

MATHEMATICAL FORMULATION

Throughout the paper we consider an elastic space (ν_2, G_2) containing a spherical inclusion (ν_1, G_1) and there is a spherical crack at their interface. Let r, θ, φ denote spherical coordinates. The crack is modeled by a mathematical cut along the part of spherical surface ($r=r_0$, $0\le\theta\le\theta_0$, $0\le\varphi\le2\pi$). The surfaces are not at contact interact and the joint is ideal along another part of the phase boundary. To solve the problem one can apply a superposition principle: the stress field outside the cut is a sum of homogeneous stressed state for a space without crack and local stressed state of the crack surfaces under uniform load of opposite sign [1]. Under such assumption the boundary problem may be formulated as follows: to solve Lame's vector equation of equilibrium under the boundary conditions:

$$\sigma_r^{(1)} = \sigma_r^{(2)}, \quad \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)}, \quad u_r^{(1)} = u_r^{(2)}, \quad u_\theta^{(1)} = u_\theta^{(2)}, \quad (r = r_0, 0 < \theta \le \pi)$$

$$\sigma_r^{(1)} = \sigma_r^{(2)} = f_1(\theta), \quad \tau_{r\theta}^{(1)} = \tau_{r\theta}^{(2)} = f_2(\theta), \quad (r = r_0, 0 < \theta \le \theta_0),$$

where $\sigma_r^{(i)}$, $\tau_{r\theta}^{(i)}$, $u_r^{(i)}$, $u_\theta^{(i)}$ – are the components of stress tensor and displacement vector for space (i=1) and spherical inclusion (i=2); $f_i(\theta)$ – are the known functions corresponding to loading on the infinity; ν_i are the Poisson's ratios, G_i are the shear modules of space (i=1) and inclusion (i=2) materials.

METHOD OF SOLUTION

The study of a problem on equilibrium of elastic bodies with cuts on surfaces of the second power is based on general solutions of the fundamental boundary problems of elasticity theory for bodies of revolution in curvilinear coordinates.

Mathematical tools applied to examine these class of tasks are very diverse. One of them is the method proposed by Love [2]. In Love's approach, the solution to the axisymmetric problem in classical elasticity may be represented in terms of a single strain function that satisfies the biharmonic equation. The method of eigenfunctions is another general one. The solutions of boundary problems for bodies of revolution in vector formulation with the help of complete vector eigenfunctions sets were given by Ulitko [1]. In the axisymmetric case the method of direct integration is the most simple and clear one for solving of Lame's equilibrium equation in the arbitrary curvilinear coordinates of revolution.
As assumed relationships the general solutions of elastic problems for a spherical inclusion and a space with a spherical cavity presented in the form of series expansions by Legendre functions were taken [3]. The problem is reduced to an interrelated system of coupled integral equations with respect to the Legendre functions, and then to a system of singular integral equations with respect to two unknown functions.
As the result the asymptotic expressions for stress components were obtained in the form [4]:

$$\sigma_{r\theta} + i\sigma_r \approx (K_2 + iK_1)\, r_0^{-0,5}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}+i\lambda} \cdot \left(\sin\frac{\theta}{2} + \sin\frac{\theta_0}{2}\right)^{-\frac{1}{2}-i\lambda},$$

where the intensity factors of normal K_1 and tangential K_2 stress are determined by formula

$$K_2 + iK_1 = -4a_0\sqrt{1-4\gamma^2}\,\sqrt{r_0}\,L(\theta_0)\cdot\frac{\Gamma(1+i\lambda)\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma(3/2)}.$$

SOME EXAMPLES

As an example let us examine the case when the cut surfaces are under normal internal pressure of intensity q. First of all note that the dependence of the stress intensity factors (SIF) on the Poisson's ratios that take values in the interval $0{,}2 \le \nu_1, \nu_2 \le 0{,}42$ is not significant. For example at $\nu_1=\nu_2=1/3$ and $\nu_1=1/3$, $\nu_2=1/4$ ($G_1=G_2$) SIF K_1 and K_2 are equal to 0,457 and 0,462, respectively. The behaviour of stress intensity factors in dependence on the ratio of inclusion and matrix shear modules $\beta=G_1/G_2$ for different angle θ_0 values of the cut is submitted in Figure 1. The results presented allows to compare stress intensity factors for a composite ($\beta>1$) and homogeneous ($\beta=1$) materials. It follows from Figure 1 that SIF increase with β increasing and exceed the corresponding values of SIF for homogeneous material. If one introduces parameters $S_1=K_1^{(k)} / K_1^{(0)}$, $S_2=K_2^{(k)} / K_2^{(0)}$, the superscripts (k) and (0) correspond to composite and homogeneous materials) they will vary in the interval $1<S_i<1{,}5$ (i=1,2) while $1<\beta<60$.

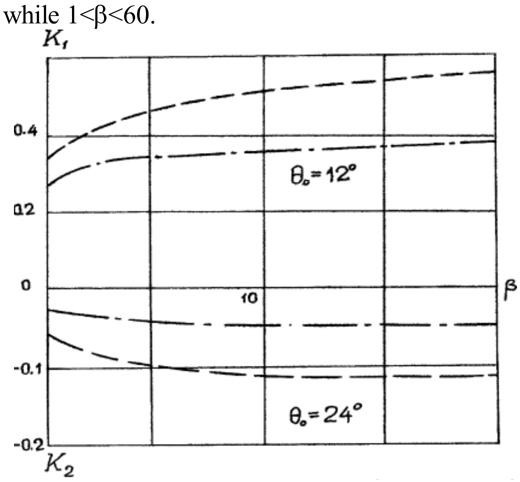

Figure 1. The dependence of stress intensity factors on the ratio of inclusion and matrix shear modules $\beta=G_1/G_2$ for different angle θ_0 values of the cut.

References

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