NONLINEAR RADIAL OSCILLATIONS OF ANISOTROPIC THIN-WALLED CYLINDRICAL TUBES

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Summary Nonlinear radial oscillations of thin-walled cylindrical tubes with either radial, tangential or longitudinal isotropy are studied. Free oscillations are considered and the effect of the anisotropy is analysed. To investigate solutions with time dependent net applied pressures the Lie point symmetry structure of the differential equations is examined.

A hyperelastic material is said to prossess transverse isotropy with respect to a direction **h** if its strain-energy function W is invariant under rotations about **h** [1,2]. Nonlinear radial oscillations of a cylindrical tube with either radial, trangential or longitudinal transverse isotropy are considered [3,4].
Consider a thin-walled anisotropic cylindrical tube of incompressible elastic material of constant density ρ^*. The inner and outer radii of the undeformed tube are ρ_1 and ρ_2 and the inner radius of the deformed tube at time t is $r_1(t)$. Let

$$x(t) = \frac{r_1(t)}{\rho_1} \ . \tag{1}$$

Pressures $P_1(t)$ and $P_2(t)$ are applied to the inner and outer surfaces. The strain-energy function, W, is of the form

$$W = W(I_1, I_2,\ K_1, K_2), \tag{2}$$

where the principal invariants, I_1 and I_2, are given by

$$I_1 = I_2 = x^2 + \frac{1}{x^2} + 1 \tag{3}$$

and K_1 and K_2 for radial, tangential and longitudinal transverse isotropy are respectively,

$$K_1 = \frac{1}{x^2}\ ,\ K_2 = \frac{1}{x^4}\ ;\ K_1 = x^2,\ K_2 = x^4\ ;\ K_1 = K_2 = 1\ . \tag{4}$$

It is shown that for radial, tangential and longitudinal transverse isotropy,

$$\frac{d^2x}{dt^2} + \frac{dW_0}{dx} = x\,P(t)\ , \tag{5}$$

where

$$W_0 = \frac{1}{\rho^*\,\rho_1^2}\,W\ , \quad P(t) = \frac{2\Big(P_1(t) - P_2(t)\Big)}{\mu\rho^*\,\rho_1^2}\ ,\ \mu = \Big(\frac{\rho_2}{\rho_1}\Big)^2 - 1\ . \tag{6}$$

Equation (5) has the same form as that for radial oscillations of a thin-walled isotropic cylindrical tube.
The strain-energy function

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + C_3(K_1 - 1) + C_4(K_2 - 1), \tag{7}$$

where C_1, C_2, C_3 and C_4 are constants, is considered. The undeformed body is stress free provided $C_3 = -2C_4$. We will assume that $C_1 + C_2 + C_4 > 0$ to satisfy stability conditions. Then for radial, tangential and longitudinal transverse isotropy, equation (5) becomes, respectively,

$$\frac{d^2x}{dt^2} + \big(D_1 - P(t)\big)x = (D_1 - D_2)\,\frac{1}{x^3} + D_2\,\frac{1}{x^5}\ , \tag{8}$$

$$\frac{d^2x}{dt^2} + \big(D_1 - D_2 - P(t)\big)x = D_1\,\frac{1}{x^3} - D_2\,x^3\ , \tag{9}$$

$$\frac{d^2x}{dt^2} + \big(D_1 - P(t)\big)x = D_1\,\frac{1}{x^3}\ , \tag{10}$$

where

$$D_1 = \frac{2(C_1 + C_2)}{\rho^*\,\rho_1^2}\ , \quad D_2 = \frac{4C_4}{\rho^*\,\rho_1^2}\ . \tag{11}$$

When $P(t)$ is constant, equations (8), (9) and (10) can be integrated immediately and the solution reduced to a quadrature. Free oscillations with initial conditions

$$x(0) = x_0 \ , \quad \dot{x}(0) = 0 \ , \tag{12}$$

are considered. For both radial and tangential transverse isotropic materials the period of oscillation is less than that for isotropic materials. The range of oscillation is $x_0 \leq x \leq x_1$ if $x_0 < 1$ and $x_1 \leq x \leq x_0$ if $x_0 > 1$. For isotropic materials $x_0 x_1 = 1$. For radial transverse isotropy it is found that $x_0 x_1 > 1$ while for tangential transverse isotropy $x_0 x_1 < 1$.

The Lie point symmetries of (8), (9) and (10) are investigated to examine if the equations can be integrated when $P = P(t)$.

Equation (10) is the Ermakov-Pinney equation which also describes radial oscillations of an isotropic cylindrical tube. It has three Lie point symmetries which can be used to derive a nonlinear superposition principle, [5,6,].

Equation (8) has no Lie point symmetries when $P = P(t)$ if $D_1 \neq D_2$. However, if $D_1 = D_2$ then (8) has the Lie point symmetry

$$X = (t + A) \, \frac{\partial}{\partial t} + \frac{1}{3} \, x \, \frac{\partial}{\partial x} \ , \tag{13}$$

provided $P(t)$ is of the form

$$P(t) = D_1 - \frac{B}{(t + A)^2} \ , \tag{14}$$

where $A > 0$ and B are arbitrary constants. Using the Lie point symmetry (13), equation (8) can be transformed to the autonomous equation

$$\frac{d^2 x^*}{dt^{*2}} - \frac{1}{3} \, \frac{dx^*}{dt^*} + \Big(B - \frac{2}{9}\Big) x^* = \frac{D_1}{x^{*5}} \ , \tag{15}$$

where

$$x^* = \frac{x}{(t + A)^{1/3}} \ , \quad t^* = \ln \left(1 + \frac{t^*}{A} \right) \ . \tag{16}$$

Equation (9) has no Lie point symmetries when $P = P(t)$ if $D_1 \neq 0$ and $D_2 \neq 0$. For the extreme anisotropic case in which $D_1 = 0$, $D_2 \neq 0$, equation (9) has the Lie point symmetry

$$x = (t + A) \, \frac{\partial}{\partial t} - x \, \frac{\partial}{\partial x} \ , \tag{17}$$

provided $P(t)$ is of the form

$$P(t) = -D_2 - \frac{B}{(t + A)^2} \ , \tag{18}$$

where $A > 0$ and B are arbitrary constants. Using the Lie point symmetry (17), equation (9) can be transformed to the autonomous equation

$$\frac{d^2 x^*}{dt^{*2}} - 3 \, \frac{dx^*}{dt^*} + (2 + B) \, x^* = -D_2 x^{*3} \ . \tag{19}$$

Equations (15) and (19) are solved numerically using a 4th order Runge-Kutta method. The solutions are compared with radial oscillations of an isotropic thin-walled cylindrical tube with the same values of D_1, D_2 and $P(t)$.

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