Localized Necking Criterion Based on Acoustic Tensor for Materials with Anisotropic Damage

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Extended Summary

Criterion for localized necking in strain-softening materials is studied. The material is supposed to have anisotropic damage and anisotropic plasticity. The critical condition of damage evolution is of primary interest in this study. For localized necking as a consequence of plastic instability, the singularity of acoustic tensor is taken as the critical condition for localized necking in strain-softening materials.

A damage-coupled general criterion for localized necking based on acoustic tensor is proposed as

$$\det(\mathbf{n}\cdot\overline{\mathbf{C}}^{ep}\cdot\mathbf{n})=0 \tag{1}$$

where $\overline{\mathbf{C}}^{ep}$ is effective tangent modulus tensor and $\mathbf{n}$ the unit normal vector of localized band.

$$\dot{\boldsymbol{\sigma}}=\overline{\mathbf{C}}^{ep}:\dot{\boldsymbol{\varepsilon}} \tag{2}$$

Based on anisotropic damage theory, we have

$$\overline{\mathbf{C}}^{ep}=\mathbf{M}^{-1}:\mathbf{A}^{-1}:\mathbf{C}^{ep}:\mathbf{M}^{T,-1} \tag{3}$$

where $\mathbf{M}$ is damage effective tensor and $\mathbf{C}^{ep}$ the tangent modulus tensor.

$$\mathbf{M}(\mathbf{D})=\begin{bmatrix}\frac{1}{1-D_1} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{1-D_2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{1-D_3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_3)(1-D_1)}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_1)(1-D_2)}}\end{bmatrix} \tag{4}$$

$$\mathbf{C}^{ep}=\mathbf{C}^{e}-\beta(\mathbf{C}^{e}:\frac{\partial F_p}{\partial\boldsymbol{\sigma}}):(\frac{\partial F_p}{\partial\boldsymbol{\sigma}^T}:\mathbf{C}^{e}) \tag{5}$$

and

$$\mathbf{A}=\mathbf{I}+\alpha\{\frac{\partial\mathbf{M}}{\partial\mathbf{D}^T}:\frac{\partial F_d}{\partial(-\mathbf{Y})}:\mathbf{M}^{-1}+[\mathbf{C}^{ep}:\mathbf{M}^{-1}:(\frac{\partial\mathbf{M}}{\partial\mathbf{D}^T}:\frac{\partial F_d}{\partial(-\mathbf{Y})})^T]:\mathbf{C}^{e^{-1}}\}:\boldsymbol{\sigma}\frac{\partial F_d}{\partial\boldsymbol{\sigma}^T} \tag{6}$$

In eqns (5) and (6), F_p is plastic yield surface and F_d the damage surface.

$$F_p(\overline{\boldsymbol{\sigma}},p)=\overline{\boldsymbol{\sigma}}_{eq}-[T_0+T(p)]=0 \tag{7}$$

and

$$F_d=Y_{eq}-[C_0+C(Z)]=0 \tag{8}$$

The damage evolution equations are

$$\dot{\mathbf{D}}=\dot{Z}\frac{\partial F_d}{\partial(-\mathbf{Y})} \tag{9}$$

and

$$\dot{Z}=\alpha\frac{\partial F_d}{\partial\boldsymbol{\sigma}^T}:\dot{\boldsymbol{\sigma}} \tag{10}$$

By deploying eqn (1), critical condition of localized necking for both negative and positive strain ratio can be obtained. For negative strain ratio, the localized necking criterion is

$$(\overline{C}_{12}^{ep}\overline{C}_{21}^{ep}+\overline{C}_{66}^{ep}\overline{C}_{21}^{ep}+\overline{C}_{66}^{ep}\overline{C}_{12}^{ep}-\overline{C}_{11}^{ep}\overline{C}_{22}^{ep})-2\overline{C}_{66}^{ep}\sqrt{\overline{C}_{11}^{ep}\overline{C}_{22}^{ep}}=0 \tag{11}$$

while the inclination angle of localization band is

$$tg\theta=\pm(\frac{\overline{C}_{11}^{ep}}{\overline{C}_{22}^{ep}})^{\frac{1}{4}} \tag{12}$$

For positive strain ratio, the localized necking criterion is

$$\overline{C}_{11}^{ep} = 0 \tag{13}$$

In eqn (3), let

$$\mathbf{\Phi} = \mathbf{C}^{ep} : \mathbf{M}^{T,-1} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & 0 \\ 0 & 0 & \Phi_{66} \end{bmatrix} \tag{14}$$

and

$$\mathbf{\Psi} = \mathbf{A} : \mathbf{M} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 \\ \Psi_{21} & \Psi_{22} & 0 \\ 0 & 0 & \Psi_{66} \end{bmatrix} \tag{15}$$

Then

$$\overline{\mathbf{C}}^{ep} = \mathbf{\Psi}^{-1} : \mathbf{\Phi} \tag{16}$$

Substitute eqns (14)-(16) into eqn (11). The localized necking criterion for negative strain ratio is revised as

$$\Psi_{66}(\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}) + [\Psi_{21}\Phi_{11} - \Psi_{22}\Phi_{12} - \Psi_{11}\Phi_{21} + \Psi_{12}\Phi_{22} + 2\sqrt{(\Psi_{12}\Phi_{21} - \Psi_{22}\Phi_{11})(\Psi_{21}\Phi_{12} - \Psi_{11}\Phi_{22})}]\Phi_{66} = 0 \tag{17}$$

The inclination angle of localized band is

$$tg\,\theta = \pm\left(\frac{\Psi_{11}\Phi_{22} - \Psi_{12}\Phi_{21}}{\Psi_{22}\Phi_{11} - \Psi_{21}\Phi_{12}}\right)^{\frac{1}{4}} \tag{18}$$

Substitute eqns (14)- (16) into eqn (13). The localized necking criterion for positive strain ratio is

$$\Psi_{22}\Phi_{11} - \Psi_{21}\Phi_{12} = 0 \tag{19}$$

In uniaxial loading condition, $\sigma_2 = \sigma_6 = 0$ and $D_2 = \eta D_1$.

$$\Phi = \begin{bmatrix} \dfrac{(1-D_1)e_0[4T'(p)+h^2e_0]}{4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0} & \dfrac{2(1-\eta D_1)e_0[2T'(p)\nu+he_0]}{4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0} & 0 \\ \dfrac{2(1-D_1)e_0[2T'(p)\nu+he_0]}{4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0} & \dfrac{4(1-\eta D_1)e_0[T'(p)+e_0]}{4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0} & 0 \\ 0 & 0 & \dfrac{\sqrt{(1-D_1)(1-\eta D_1)}e_0}{2(1+\nu)} \end{bmatrix} \tag{20}$$

and

$$\Psi_{11} = \frac{1}{1-D_1} + \frac{4T^2\{2T'(p)[2-(1+\eta)\nu^2]+[2+h^2-h(2+\eta)\nu]e_0 - \eta D_1[4T'(p)(1-\nu^2)+(2+h^2-3h\nu)e_0]\}}{(1-\eta D_1)(1-D_1)[4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0][3T^2-2C'(Z)e_0(1-D_1)^2]} \tag{21}$$

$$\Psi_{12} = \frac{2T^2\nu(1-D_1)(2\eta D_1-1-\eta)\{2T'(p)[2-(1+\eta)\nu^2]+[2+h^2-h(2+\eta)\nu]e_0 - \eta D_1[4T'(p)(1-\nu^2)+(2+h^2-3h\nu)e_0]\}}{(1-\eta D_1)^4[4T'(p)(-1+\nu^2)-(4+h^2-4h\nu)e_0][3T^2-2C'(Z)e_0(1-D_1)^2]}$$

$$\Psi_{21} = \frac{4T^2[2T'(p)(-1+\eta)\nu-(h-2\eta\nu)e_0+\eta(h-2\nu)D_1e_0]}{(1-\eta D_1)(1-D_1)[4T'(p)(1-\nu^2)+(4+h^2-4h\nu)e_0][3T^2-2C'(Z)e_0(1-D_1)^2]}$$

$$\Psi_{22} = \frac{1}{1-\eta D_1} - \frac{2(1-D_1)T^2\nu(1+\eta-2\eta D_1)[2T'(p)(1-\eta)\nu+(h-2\eta\nu)e_0-\eta(h-2\nu)D_1e_0]}{(1-\eta D_1)^4[4T'(p)(-1+\nu^2)-(4+h^2-4h\nu)e_0][3T^2-2C'(Z)e_0(1-D_1)^2]}$$

$$\Psi_{66} = \frac{1}{\sqrt{(1-D_1)(1-\eta D_1)}}$$

Substituting eqns (20) and (21) into eqn (17), we can get an equation about D_1, i.e.

$$f(D_1, T, T'(p), C'(z)) = 0 \tag{22}$$

By solving eqn (22), we can get the critical damage value D_{1c} in terms of T, $T'(p)$ and $C'(z)$ for negative strain ratio.

For eqiaxial tension, $\sigma_1 = \sigma_2$ and $D_1 = D_2$. Similar equation as eqn (22) can be obtained for the solution of critical damage value D_{1c} at the positive strain-ratio region.

So far, the anisotropic damage model is coupled with the localized necking criterion based on acoustic tensor. Closed-form expressions for localized necking criteria for both positive and negative strain ratio are presented. Both of the criteria and the inclination angle are in relations with current stress and strain states as well as damage variables, which indicates that critical damage value for localized necking is not a constant. The proposed criterion is applicable to non-proportional loading condition involving multistage stamping processes. The closed-form expression of localized necking criterion has application potential in the analysis of failure in strain-softening materials such as hot metals, rocks, soil, solder, etc.