OPTIMAL ROBUST $H_{\infty}$-CONTROL OF MECHANICAL SYSTEMS

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Summary A survey of contemporary approaches to robust $H_{\infty}$-controllers design for optimal attenuating oscillations in uncertain mechanical systems is given. There are two basic approaches to solving this problem. One of them is based on solving Riccati equations, while another approach is based on linear matrix inequality technique. It is shown that Riccati equations associated to this problem and containing additional parameters, Lagrange multipliers, may be feasible only when values of these parameters belong to a parallelepiped boundaries of which are identified. It is also shown that robust $H_{\infty}$-controller design can be reduced to solving an optimization problem subject to LMI constraints. Effective algorithms implemented in MATLAB are suggested. As an illustrative example, the problem of optimal attenuating oscillations for a parametrically disturbed pendulum is considered.

INTRODUCTION

There are two main approaches to $H_{\infty}$-controller design in framework of state-space technique. One of them is based on solving algebraic Riccati equation, while another one on solving linear matrix inequalities (LMIs). When a mathematical model of the plant to be controlled is completely known each of these approaches allows to synthesize $H_{\infty}$-controllers by using effective computational procedures of MATLAB. In the case when a description of the plant involves uncertainty, synthesis of robust $H_{\infty}$-controllers providing a desired disturbance attenuation is reduced to standard $H_{\infty}$-control design for some auxiliary system. This completely known system comes from the initial uncertain system by means of replacement uncertainty for some additional artificial disturbances which enter system equations with additional parameters. The synthesis of $H_{\infty}$-controllers for the auxiliary system is very difficult problem because the values of the additional parameters for which appropriate Riccati equations or LMIs are feasible are unknown a priori. In recent paper [1], it was proved that the domain of admissible values of the admissible parameters, where $H_{\infty}$-control problem for the auxiliary system is feasible, is bounded and included in a parallelepiped boundaries of which were calculated. This result based on analysis of Riccati equations and limiting control possibilities allows to derive an estimate for the robust performance in the initial uncertain system. It was also shown [2] that robust $H_{\infty}$-controller design can be reduced to an optimization problem for some function subjected to LMIs constraints which contain also additional parameters as variables. It turns out that in the state feedback case, this optimization problem is convex, whereas in the output feedback case this problem is non-convex in principle. The algorithm proposed in [2] is an iterative process on each iteration of which a minimum of a linear function subjected to LMIs constraints is found by using the command mincx in MATLAB. These approaches can be applied to various mechanical systems with uncertainty. Effectiveness of the algorithms is demonstrated on the example of a parametrically disturbed pendulum.

STATEMENT OF THE PROBLEM

Consider a controlled mechanical system described by the equations

$$
\begin{align*}
\dot{x} &= Ax + B_1 v + B_2 u , \\
z &= C_1 x + D_{12} u , \\
y &= C_2 x + D_{21} v ,
\end{align*}
$$

(1)

where $x$ is a state, $v$ is disturbance input, $u$ is control input, $z$ is controlled output, $y$ is measured output. It is assumed that the matrix $A$ in these equations is presented in the form

$$
A = A_0 + \sum_{k=1}^{n} F_k \Omega_k(t, x) E_k ,
$$

(2)

where $A_0$ is a given matrix, $F_k$, $E_k$ are given matrices and $\Omega_k(t, x)$ are unknown matrix functions satisfying inequalities

$$
\Omega_k(t, x)^T \Omega_k(t, x) \leq I , \quad \forall t \geq 0 , \quad \forall x , \quad k = 1, \ldots, n .
$$

(3)

Such a structure of the matrix $A$ allows to describe uncertainty in different matrix entries and blocks. Denote a class of such uncertainties through $\Sigma$. The problem is to synthesize a linear dynamic output controller providing fulfillment of the inequality

$$
\| z \| < \gamma \| v \| , \quad \forall v \in L_2 , \quad v \neq 0 , \quad \forall \Omega_k(t, x) \in \Sigma
$$

(4)

with a minimally possible value of $\gamma$. 

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MAIN RESULTS

The main idea of synthesizing robust controller for the uncertain system under consideration is to replace the initial uncertain system by some auxiliary system without uncertainty, including additional disturbance inputs and additional parameters. This auxiliary system will be of the form

\[ \dot{x} = A_0 x + (B_1 \gamma \mu_1^{-1} F_1 \cdots \gamma \mu_n^{-1} F_n) \xi + B_2 u \]

\[ \dot{\xi} = \left( \begin{array}{c} C_1 \\ \mu_1 E_1 \\ \vdots \\ \mu_n E_n \end{array} \right) x + \left( \begin{array}{c} D_{12} \\ 0 \\ \vdots \\ 0 \end{array} \right) u \]

where \( \xi = \text{col} (\xi_0, \xi_1, \ldots, \xi_n) \) is a disturbance input, \( \dot{\xi} \) is a controlled output, all matrices in these equations are the same as those in the equations (1), (2), \( \gamma > 0 \) and \( \mu_k > 0, k = 1, \ldots, n \) are some numbers. The control law providing for the system (5) with some values of \( \mu_k > 0, k = 1, \ldots, n \) fulfillment of the inequality

\[ \| \dot{\xi} \| < \gamma \| \xi \|, \quad \forall \xi \in L_2, \quad \xi \neq 0, \]

provides also fulfillment the inequality (4) for the initial uncertain system (1), (2) with the same value of \( \gamma \).

**Theorem.** Let for the system (5) in which \( A_0 \) is Hurwitz and \( D_{12}^T (C_1, D_{12}) = (0, I), D_{21}(B_1^T D_{12}^T) = (0, I) \), the problem of output \( H_\infty \)-control for a given \( \gamma \) and some \( \mu = (\mu_1, \ldots, \mu_n) \) be feasible. Then the values of \( \mu \) belong to the open parallelepiped

\[ \Pi(\gamma) = \{ \mu_k \in (\mu_k^-, \mu_k^+), k = 1, \ldots, n \}, \]

\[ \mu_k^- = \frac{\| Q_k \|_\infty}{\sqrt{1 + \| H_{12} \|_\infty^2}}, \quad \mu_k^+ = \frac{\sqrt{1 + \| H_{21} \|_\infty^2}}{\| R_k \|_\infty}, \]

and

\[ H_{12}(s) = C_1(s I - A_0)^{-1} B_2, \quad H_{21}(s) = C_2(s I - A_0)^{-1} B_1, \]

\[ Q_k(s) = C_1(s I - A_0)^{-1} F_k, \quad R_k(s) = E_k(s I - A_0)^{-1} B_1. \]

Alternatively, an optimization algorithm for checking feasibility of the robust \( H_\infty \)-control problem based on solving LMIs is suggested. This algorithm is implemented by means of the standard minimization procedure for a linear function under LMI constraints (command mincx in the LMI toolbox). It is shown that the sequence of the function values generated by the algorithm is always converging. If its limiting value is equal to zero, the original robust \( H_\infty \)-control problem is feasible.

**OPTIMAL ATTENUATION OF OSCILLATIONS FOR A PARAMETRICALLY DISTURBED PENDULUM**

Consider a controlled pendulum under parametric and external disturbances described by the equation

\[ \ddot{\varphi} + [1 + f_1 \Omega_1(t, \varphi, \dot{\varphi})] \dot{\varphi} + \omega_0^2 [1 + f_2 \Omega_2(t, \varphi, \dot{\varphi})] \varphi = u + v_1 \]

\[ \psi = \varphi + v_2, \quad \varphi(0) = \dot{\varphi}(0) = 0 \]

\[ z = \text{col} (\omega_0 \varphi, \dot{\varphi}, u) \]

where \( \omega_0, f_i, i = 1, 2 (\omega_0 \neq 0, 0 \leq f_i < 1) \) are given parameters. It is required to synthesize a controller using the measured output \( \psi \) and providing the optimal attenuation of the external and measurement disturbances. The pendulum parameters were chosen as follows: \( \omega_0 = 10 \) and \( f_1 = f_2 = 0.1 \). The parallelepiped (6) is given by

\[ 0.0817 < \mu_1 < 1.0055 \gamma, \quad 0.0817 < \mu_2 < 0.1005 \gamma. \]

The appropriate optimal controller provides attenuating oscillations for the parametrically disturbed pendulum in the ratio of no more than 15.5. For comparison, note that the minimal \( H_\infty \)-performance for the pendulum without parametric disturbances is equal to 1.41.

**References**
