CHARACTERISATION OF THE CYCLIC BEHAVIOUR OF ELASTIC-PLASTIC-CREEPING BODIES

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Abstract The paper describes a systematic theoretical approach to the characterization of the cyclic behaviour of a body that exhibits creep and plasticity. The cyclic state for a particular constitutive equation may be characterized as a minimum theorem in terms of a class of kinematically admissible inelastic strain rate history. The minimum for some particular class of approximating strain rate histories my be determined by means of a new mathematical programming method, the Linear Matching Method, that sequentially reduces the functional through the solution of linear problems. The theory is developed for the standard Mandel thermodynamic model of constitutive behaviour allowing for both plasticity and creep behaviour. Examples are given of applications typical of those occurring in the life assessment of high temperature structures where the load corresponding to a particular failure condition is required.

EXTENDED SUMMARY

INTRODUCTION

Classical plasticity and creep theory was primarily developed to assist in the design of engineering structures. Design codes and life assessment procedures used on a routine basis in industry rely, at their core, on concepts such as limit load, shakedown limit, reference stress and elastic follow-up all derive from basic plasticity theory and its extensions to creep behaviour. Generally these procedures rely upon linear elastic solutions and a sequence of rule based assessment. Over the last 20 years computational methods using advanced constitutive equations have significantly replaced ruled based methods in design. However, unlike rule-based methods, they fail to directly provide answer to the type of question posed by design and life assessment. For example, for the assessment of the strength of a structure, the designer wishes to know the maximum load or load range for which a certain structural phenomenon may occur. Such phenomena may include plastic collapse, excessive growth of creep strain, creep rupture at a particular time, initiation of failure due creep/fatigue interaction all of which may be regarded as related to properties of the inelastic strain rate history.

In recent years there has begun a reassessment of theoretical and computational methods that are capable of providing such direct answers to design related issues. Most of this work has been concerned with shakedown limits [8] but there is a need for methods capable of relating to a wide range of low and high temperature structural phenomenon. For this to be achieved, there is a need for a consistent theoretical approach, combined with appropriate approximation procedures that may be applied to a much wider range of constitutive assumptions than the simpler plasticity models. This forms the subject of this paper. The approach consists of two parts. The first is concerned with the characterization of the cyclic state as a minimum theorem; the second involves strategies for determining the minimum for classes of approximating inelastic strain rate histories. The purpose of the paper is to describe the theoretical basis for these two phases for a class of creep and plasticity constitutive relationships that conform to the Mandel thermodynamic theory, and to present examples of applications.

MINIMUM PRINCIPLES FOR CYCLIC BEHAVIOUR

Consider first the simplest classical case of a body subjected to small deformations, a linear elastic perfectly plastic material and a cyclic history of load $\lambda P_i(x_i,t)$ and temperature $\lambda\theta(x_i,t)$ where λ denotes a load parameter. For certain classes of constitutive relationship, a cyclic state of response exists during a typical cycle, , $0 \le t \le \Delta t$, where the stress history is given by;

$$\sigma_{ij}(x_i,t) = \lambda\hat{\sigma}_{ij} + \bar{\rho}_{ij}(x_i) + \rho^r_{ij} , \qquad \hat{\sigma}_{ij} = \hat{\sigma}^P_{ij} + \hat{\sigma}^\theta_{ij} \tag{1}$$

Here $\hat{\sigma}^P_{ij}$ and $\hat{\sigma}^\theta_{ij}$ denote the linear elastic solutions corresponding to $P_i(x_i,t)$ and $\theta(x_i,t)$ respectively, $\bar{\rho}_{ij}$ denotes a time constant residual stress field, corresponding to the residual stress field at the beginning and the end of the cycle, and $\rho^r_{ij}(x_i,t)$ denotes the change in the residual stress field during the cycle, with $\rho^r_{ij}(x_i,0) = \rho^r_{ij}(x_i,\Delta t) = 0$. For a prescribed load parameter λ the exact cyclic solution may be characterized by the following minimum theorem [1].

A kinematically admissible plastic strain rate history is defined as any history $\dot{\varepsilon}^c_{ij}$ for which the accumulated strain over a cycle $\Delta\varepsilon^c_{ij} = \int_0^{\Delta t} \dot{\varepsilon}^c_{ij} dt$ is compatible with a displacement increment Δu^c_i that satisfies displacement boundary conditions. Then the exact cyclic solution $\dot{\varepsilon}^c_{ij} = \dot{\varepsilon}^p_{ij}$ minimizes the functional,

$$I(\dot{\varepsilon}^c_{ij},\lambda) = \int_0^{\Delta t}\int_v (\sigma^c_{ij} - \lambda\hat{\sigma}_{ij})\dot{\varepsilon}^c_{ij} dV dt \tag{2}$$

subject to the condition that there exists a constant residual stress field $\bar{\rho}_{ij}$ so that the stress history;

$$\sigma^{*}_{ij}(x_i,t) = \lambda\hat{\sigma}_{ij} + \bar{\rho}_{ij}(x_i) + \rho^{c}_{ij} \qquad (3)$$

satisfies the yield condition during the cycle. Here σ^{c}_{ij} denotes the stress at yield associated with $\dot{\varepsilon}^{c}_{ij}$ and ρ^{c}_{ij} denotes the history of residual stress corresponding to $\dot{\varepsilon}^{c}_{ij}$. Both of the classical shakedown theorems may be derived as special cases when the magnitude of $\dot{\varepsilon}^{c}_{ij}$ is assumed to be sufficiently small for ρ^{c}_{ij} to be negligible small in (3). The result is closely related to an earlier result by Gokhfeld and Cherniavsky [2].

For more general constitutive relationships this result may be generalized. For a viscous material, as a model of high temperature creep, the functional I becomes,

$$I(\dot{\varepsilon}^{c}_{ij},\lambda) = \int_{0}^{\Delta t}\int_{v} \Omega(\dot{\varepsilon}^{c}_{ij}) - \lambda\hat{\sigma}_{ij}\dot{\varepsilon}^{c}_{ij} dVdt \qquad (4)$$

where $\Omega(\dot{\varepsilon}^{c}_{ij})$ is a flow potential [4,5]. A further generalization may be made for constitutive relationships that conform to the Mandel thermodynamic theory [6,7] where the behaviour, for a single state variable A, may be expressed in terms of a Helmholtz free energy $\phi(A)$ and a flow potential $\Omega(\dot{\varepsilon}^{c}_{ij}, A)$. Such generalizations have been discussed by Pollizzotto [3] for time independent plasticity and may be generalized to creep behaviour. The full paper and the presentation will provide a discussion of these generalizations.

THE LINEAR MATCHING METHOD

Non linear programming methods have been applied to shakedown problems for many years, following early work by Maier and others. A recent summary indicates the diversity of approaches that are now possible [8]. A particularly simple and flexible method has emerged in recent times, based in concept on methods used for many years in engineering design [8,9]. When applied to the minimization of the functionals (2) and (4), the method essentially consists of the solution of a sequence of linear problems where the linear material coefficients are evaluated from two matching conditions. For a known initial kinematically admissible strain rate history $\dot{\varepsilon}^{i}_{ij}$ the state of stress given by the linear material and the constitutive assumptions are matched. In addition the condition of kinematical admissibility is imposed on the solution of the linear problem. The solution of the linear problem then provides a new kinematically admissible strain rate history that reduces the functional (2) or (4), provided certain convexity conditions are satisfied. During the iterative process, it is then possible to adjust the load parameter λ so that a restriction on some property of the strain rate history is maintained. This methodology has been applied to a range of problems associated with the life assessment of structures [8,9] and has begun to form the basis for methods for use in industry.
The application of such methods to constitutive relationships involving internal state variable requires an extension to the matching of the internal state. This may be done in a number of alternative ways, providing solutions with a range of assumptions concerning the relative magnitude of the timescales involved in changes in the state variable and changes in the residual stress field. This allows a systematic approach to simplification of cyclic loading problems for creep, taking into account known properties of cyclic solutions.

CONCLUSIONS

Previous work on the characterization of the cyclic behaviour of structures for plasticity and creep has identified minimum theorems in terms of kinematically admissible strain rate histories. A new programming method, the Linear Matching Method, has demonstrated that the identification of the minimum for approximating classes of such fields may be achieved for complex problems of the type that occur in the life assessment of structures. The paper gives a systematic account of both the minimum theorems and the Linear Matching Method for plasticity and creep constitutive relationships expressed in terms of the standard Mandel thermodynamic model. Examples of applications show that it is possible to evaluate the loads corresponding to particular failure modes, interpreted as properties of the strain rate history. Such solutions are of particular value for industrial applications.

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