OPTIMAL FORCE ACTION AND REACTION IN STRUCTURAL DESIGN AND IDENTIFICATION

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Summary. Two classes of optimization problems are discussed in the paper: optimal loading distribution providing maximal or minimal structural response and optimal loading for structural identification providing maximum of the distance measure between the computed response of a model and the response of the actual structure measured in an experiment. Various measures of structural response are discussed, namely the total potential energy which represents the global stiffness of a structure, quadratic norm of displacement vector or arbitrary functional expressed in displacements. Derived optimality conditions for the optimal loading have the form of the coaxiality rule between forces and displacements and/or follow from the solution of the respective eigenvalue problems. The theory is illustrated by examples of optimal loading in structural problems and identification.

OPTIMAL FORCE ACTION

Problems of optimal reaction forces were first formulated in the form of maximization of global stiffness expressed by the total potential energy Π or maximization of limit load [1]. This approach was used in [2] for optimization of loading. Opposite formulation was proposed in [3], where the optimal loading provided minimum Π. Thus obtained the "worst" loading was used for optimal design of composites. The idea was further developed in [7], where it was shown that the formulation leads to Stekloff eigenvalue problem. The present paper is concerned with a generalized class of problems, where the optimal loading distribution provides maximal or minimal structural response measured by various response functions. Discrete formulation will be used.

Let us consider the general case when a structure is subjected to a constant load $\mathbf{f}_0$ and then the load $\Delta\mathbf{f}$ is superposed on it. The designer's concern is to find the best or the worst superposed load $\Delta\mathbf{f} = \mathbf{f} - \mathbf{f}_0$, which is associated with extremal structural response to the total load $\mathbf{f}$. Let us introduce the structure response measure in the form of the general function of displacements $\mathbf{u}$, namely $\mathrm{I} = \mathrm{I}(\mathbf{u})$ and the constraint on $\mathbf{f} - \mathbf{f}_0$ in the form of the quadratic norm. The problem can be formulated in the form: find optimal load $\mathbf{f}$, which provides extremum I and satisfies respective constraints, namely

$$\mathrm{I}(\mathbf{u}) \to \text{extremum}, \text{ subject to } \quad \mathbf{Ku} = \mathbf{f}\,, \quad (\mathbf{f}-\mathbf{f}_0)^{\mathrm{T}}(\mathbf{f}-\mathbf{f}_0) - \rho_0^2 = 0\,, \tag{1}$$

where $\mathbf{K}$ denotes the global stiffness matrix. The Lagrange function has the form

$$\mathrm{I}^{\mathrm{L}}(\mathbf{u},\mathbf{f},\mathbf{u}^{\mathrm{a}},\eta) = \mathrm{I}(\mathbf{u}) - (\mathbf{u}^{\mathrm{a}})^{\mathrm{T}}(\mathbf{Ku}-\mathbf{f}) - \frac{1}{2}\eta\left[(\mathbf{f}-\mathbf{f}_0)^{\mathrm{T}}(\mathbf{f}-\mathbf{f}_0) - \rho_0^2\right] \tag{2}$$

and the stationarity condition can be expressed as follows

$$\delta\,\mathrm{I}^{\mathrm{L}} = \delta\mathbf{u}^{\mathrm{T}}\left(\frac{\partial\,\mathrm{I}}{\partial\mathbf{u}} - \mathbf{Ku}^{\mathrm{a}}\right) + \delta\mathbf{f}^{\mathrm{T}}\left(\mathbf{u}^{\mathrm{a}} - \eta(\mathbf{f}-\mathbf{f}_0)\right) = 0\,. \tag{3}$$

Assuming $\delta\mathbf{u}$ and $\delta\mathbf{f}$ as independent vectors, the necessary stationarity conditions are

$$\mathbf{Ku}^{\mathrm{a}} = \frac{\partial\,\mathrm{I}}{\partial\mathbf{u}} = \mathbf{f}^{\mathrm{a}}(\mathbf{f}), \quad \mathbf{u}^{\mathrm{a}} = \eta(\mathbf{f}-\mathbf{f}_0)\,. \tag{4}$$

The first condition (4) specifies the adjoint problem and the second provides the *coaxiality rule* between the adjoint displacement $\mathbf{u}^{\mathrm{a}}$ and the primary load vector $\mathbf{f} - \mathbf{f}_0$. Introducing $\mathbf{u}^{\mathrm{a}} = \mathbf{K}^{-1}\mathbf{f}^{\mathrm{a}} = \mathbf{Df}^{\mathrm{a}}$ into $(4)^2$ the coaxiality condition can be expressed in explicit form, which may in general be nonlinear as

$$\mathbf{Df}^{\mathrm{a}}(\mathbf{f}) = \eta(\mathbf{f}-\mathbf{f}_0), \text{ or } \quad \mathbf{K}(\mathbf{u}-\mathbf{u}_0) = \frac{1}{\eta}\mathbf{u}^{\mathrm{a}}(\mathbf{u})\,. \tag{5}$$

Let $\mathrm{I}(\mathbf{u})$ be a quadratic function with a symmetric positive definite weighting matrix $\mathbf{A}$, thus

$$\mathrm{I}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{Au}\,, \qquad \mathbf{f}^{\mathrm{a}} = \frac{\partial\,\mathrm{I}}{\partial\mathbf{u}} = \mathbf{Au} = \mathbf{ADf} \tag{6}$$

and the equation (5) becomes linear

$$\mathbf{D}^{\mathrm{T}}\mathbf{ADf} = \eta(\mathbf{f}-\mathbf{f}_0) \quad \text{or} \quad \mathbf{Bf} = \eta(\mathbf{f}-\mathbf{f}_0)\,, \quad \mathbf{B} = \mathbf{D}^{\mathrm{T}}\mathbf{AD}\,. \tag{7}$$

In a special case we may assume $\mathbf{A} = \mathbf{1}$, then $\mathbf{f}^{\mathrm{a}} = \mathbf{Df} = \mathbf{u}$ in (6) and $\mathbf{B} = \mathbf{D}^{\mathrm{T}}\mathbf{D}$ in (7). Assumption $\mathbf{A} = \mathbf{K}$ is equivalent to the energy control which provides minimum or maximum of the global complementary energy $C = 0.5\,\mathbf{u}^{\mathrm{T}}\mathbf{Ku}$. In the latter case we arrive at optimal loading conditions $(4)^2$, $(5)^1$ and $(7)^1$ in the form $\mathbf{Df} = \mathbf{u} = \eta\,(\mathbf{f} - \mathbf{f}_0)$.

Now, let us assume that initial loading does not appear and the total load $\mathbf{f}$ can be optimally controlled. Setting $\mathbf{f}_0 = \mathbf{0}$ in the derived stationarity conditions we arrive at coaxiality rules between total force vectors $\mathbf{f}$ and adjoint displacements $\mathbf{u}^{\mathrm{a}}$ (or $\mathbf{u}$) in $(4)^2$ and (7). Moreover, the stationarity conditions $(5)^1$ and (7) take the form of eigenvalue problems. The optimal load action can be illustrated in the form of ellipsoids in the space of load parameters. The principal axes of the ellipsoids coincide with the extremal load vectors. This will be illustrated by a series of examples.

OPTIMAL LOADING IN STRUCTURAL IDENTIFICATION

Assume that the actual structure is described by the stiffness matrix $\mathbf{K}_1$, which is specified from experimental data, whereas $\mathbf{K}_2$ refers to an assumed structure model. Our aim is to find the optimal loading **f,** which maximizes the distance norm $\mathrm{I} = \Psi(\mathbf{u}_2 - \mathbf{u}_1)$

$$\mathrm{I} = \Psi(\mathbf{u}_2 - \mathbf{u}_1) \to \max\,, \quad \text{subject to} \quad \mathbf{K}_1\mathbf{u}_1 = \mathbf{f}\,,\ \ \mathbf{K}_2\mathbf{u}_2 = \mathbf{f}\,,\ \ \mathbf{f}^{\mathrm{T}}\mathbf{f} - \rho_0^2 = 0\,. \tag{8}$$

The associated Lagrangian and its variation take the form

$$\mathrm{I}^{\mathrm{L}} = \mathrm{I} - (\mathbf{u}_1^{\mathrm{a}})^{\mathrm{T}}(\mathbf{K}_1\mathbf{u}_1 - \mathbf{f}) + (\mathbf{u}_2^{\mathrm{a}})^{\mathrm{T}}(\mathbf{K}_2\mathbf{u}_2 - \mathbf{f}) - \frac{1}{2}\mu\left(\mathbf{f}^{\mathrm{T}}\mathbf{f} - \rho_0^2\right) \tag{9}$$

$$\delta\mathrm{I}^{\mathrm{L}} = \left(\frac{\partial\psi}{\partial\mathbf{u}_2}\right)^{\mathrm{T}}\delta\mathbf{u}_2 + \left(\frac{\partial\psi}{\partial\mathbf{u}_1}\right)^{\mathrm{T}}\delta\mathbf{u}_1 - \left(\mathbf{u}_1^{\mathrm{a}}\right)^{\mathrm{T}}\mathbf{K}_1\delta\mathbf{u}_1 + \left(\mathbf{u}_2^{\mathrm{a}}\right)^{\mathrm{T}}\mathbf{K}_2\delta\mathbf{u}_2 + \delta\mathbf{f}^{\mathrm{T}}\left(\mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} - \mu\mathbf{f}\right) = 0\ . \tag{10}$$

Here $\mathbf{u}_1^{\mathrm{a}}$, $\mathbf{u}_2^{\mathrm{a}}$ and μ are the Lagrange multipliers. Introduce the adjoint systems specified by

$$\mathbf{K}_1\mathbf{u}_1^{\mathrm{a}} = \frac{\partial\psi}{\partial\mathbf{u}_1} = \mathbf{f}^{\mathrm{a}}\,, \qquad \mathbf{K}_2\mathbf{u}_2^{\mathrm{a}} = -\frac{\partial\psi}{\partial\mathbf{u}_2} = +\frac{\partial\psi}{\partial\mathbf{u}_1} = \mathbf{f}^{\mathrm{a}}\,. \tag{11}$$

The optimality condition for load distribution now takes the form

$$\delta\mathbf{f}^{\mathrm{T}}(\mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} - \mu\mathbf{f}) = 0\,, \quad \text{hence} \quad \mathbf{u}_2^{\mathrm{a}} - \mathbf{u}_1^{\mathrm{a}} = \mu\mathbf{f}\,, \ \text{ or } \ (\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f}^{\mathrm{a}} = \mu\mathbf{f}\ . \tag{12}$$

Here the optimal load is coaxial to the difference of adjoint displacements of the actual structure and the model. In the case when the energy norm $\mathrm{I} = \mathrm{I}_2 = 0.5\mathbf{f}^{\mathrm{T}}(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f}$ is used as the measure of the distance between responses of the real structure and a model [3], then in (12) the adjoint displacements and forces $\mathbf{u}_1^{\mathrm{a}}$, $\mathbf{u}_2^{\mathrm{a}}$ and $\mathbf{f}^{\mathrm{a}}$ are substituted by primary vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and **f**, respectively. Then, the optimal load follows from eigenvalues of $\mathbf{D}_2$ - $\mathbf{D}_1$. Now, assume **A** to be a quadratic positive definite matrix. Thus, another distance measure $\mathrm{I}=\mathrm{I}_3$ and optimality condition can appear

$$\mathrm{I} = \mathrm{I}_3 = \frac{1}{2}(\mathbf{u}_2 - \mathbf{u}_1)^{\mathrm{T}}\mathbf{A}(\mathbf{u}_2 - \mathbf{u}_1), \quad \text{hence } (\mathbf{D}_2 - \mathbf{D}_1)^{\mathrm{T}}\mathbf{A}(\mathbf{D}_2 - \mathbf{D}_1)\mathbf{f} = \mu\mathbf{f}\,, \ \text{ or } \ \mathbf{L}\mathbf{f} = \mu\mathbf{f}\ . \tag{13}$$

The optimal loading follows from the eigenvalue problem $(13)^2$. In simple but practical case when $\mathbf{A} = \mathbf{1}$, the optimal loading can be computed as eigenvectors of the square of $\mathbf{D}_2$ - $\mathbf{D}_1$. Various distance measures were discussed in [5].

Examples of the identification will be presented, where the distance measures I_2, I_3 and the Euclidean norm of the matrix $\mathbf{D}_2$ - $\mathbf{D}_1$ will be used. Three Euclidean vector spaces are used: R^n for FEM, R^m for the searched design parameters, and R^k for the load control. A step-by-step procedure is proposed, where at each identification step the optimal load is used. Bimodal solutions were encountered in the examples. At the final stage of the identification procedure the wavelet representation of design parameters is used, too.

CONCLUDING REMARKS

Using various response functions the sensitivity derivatives with respect to load parameters and the optimality conditions were derived. The optimal load conditions have the form of the generalized coaxiality rule of loads and adjoint displacements and/or follow from the eigenvalue problems. Numerical examples of structural identification proved the usefulness of the derived optimality conditions.

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