Topology Optimization of the Geometrically Nonlinear Structures Made of Rubberlike Material

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Summary. In this paper a brief review of the topology optimization of the geometrically nonlinear plate composed of a rubberlike material is presented. Thus the material is assumed to be incompressible hyperelastic subjected to the plane stress. A demonstrative example is presented after the topics of the theory of nonlinear continuum mechanics and optimization are introduced.

INTRODUCTION

Typically most structural topology optimization cases are based on the linear elastic assumption. But investigating initially linear structures the optimized structure may be nonlinear as well the structures subjected to the large deformations. So the topics of the nonlinear continuum mechanics, aspects of FEM discretization and solving procedure of the topology optimization are presented and applied to the 2D-plate made of highly nonlinear elastic rubber.

BASIC THEORY OF NONLINEAR COMPUTATIONAL MECHANICS

Linearization and Newton-Raphson process

The basis for FEM is the spatial total virtual work of a deformable body expressed in equilibrium state as,

$$\delta W = \int_v \sigma : \delta \mathbf{d} dv - \int_v \mathbf{f}_b \cdot \delta \mathbf{v} dv - \int_{\partial v} \mathbf{t} \cdot \delta \mathbf{v} da = 0 \tag{1}$$

Above $\boldsymbol{\sigma}$ is *Cauchy* stress, $\delta\mathbf{d}$ is rate of deformation tensor corresponds to the virtual velocity field $\delta\mathbf{v}$. $\mathbf{f}_b$ are body forces per unit volume v and $\mathbf{t}$ tractions per unit surface area. The rubberlike material is typically incompressible hyperelactic and usually *Ogden, Mooney-Rivlin* or *neo-Hookean* material. Thus the homogenous elastic function may be postulated e.g. to last one as [3] , $\psi(\mathbf{C}) = \mu(\mathrm{I}_{\mathbf{C}} - 3)$, where $\mathbf{C}$ is the right Cauchy-Green strain tensor defined with deformation gradient $\mathbf{F}$ as $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ depended of the displacement field $\mathbf{u}$. $\mathrm{I}_{\mathbf{c}}$ is its first principal invariant or tr$\mathbf{C}$ and μ the material parameters determined experimental. The solution of the virtual equilibrium system (1) is achieved using N-R iterative procedure. For that the virtual work equation has to be linearized. Splitting the total virtual work into the internal and external part and further the internal part into the constitutive and initial stress components one can derive the linearized formula for internal part as follows [2]:

$D\delta W_{\mathrm{int}}[\mathbf{u}] = \int_v \delta\mathbf{d} : \mathbf{c} : \varepsilon\, dv + \int_v \sigma : [(\nabla\mathbf{u})^T \nabla\delta\mathbf{v}]\, dv$ where $\mathbf{c}$ is spatial elasticity and ε small strain tensor. The discretized formula for the constitutive component of element e with the standard shape functions N_k connected to node $k = a, b$ is as,

$D\delta W^e_{\mathrm{int}}[N_b \mathbf{u}_b] = \int_v \frac{1}{2}(\delta\mathbf{v}_a \otimes \nabla N_a + \nabla N_a \otimes \delta\mathbf{v}_a) : c : \frac{1}{2}(\mathbf{u}_b \otimes \nabla N_b + \nabla N_b \otimes \mathbf{u}_b) dv$ or rearranged in matrix form as,

$D\delta W^e_{\mathrm{int}}[N_b \mathbf{u}_b] = \delta\mathbf{v}_a \bullet \mathbf{K}_c^{(e)} \bullet \mathbf{u}_b$, where $\mathbf{K}_c^{(e)}$ is the constitutive part of elements tangent stiffness matrix whose components in familiar ortogonal *Cartesian* coordinate system are as,

$$[\mathbf{K}^{(e)}_{c,ab}]_{ij} = \int_{v^{(e)}} \sum_{k,l=1}^{2} \frac{\partial N_a}{\partial x_k} c_{ikjl} \frac{\partial N_b}{\partial x_l} dv, \quad i, j = 1,2. \tag{2}$$

Further, in general the elasticity tensor can be split into the deviatoric and pressure component and another variational approach has to be used to avoid so-called *locking* phenomenon (*multi-field variational principles*). But now in-plane stress situation e.g. the neo-Hookean elastic function takes the form [2], $\psi(\overline{\mathbf{C}}) = \frac{1}{2}\mu(\mathrm{I}_{\overline{\mathbf{C}}} + \mathrm{III}^{-1}_{\overline{\mathbf{C}}} - 3)$. $\overline{\mathbf{C}}$ is the in-plane part of $\mathbf{C}$. By relation $\mathbf{c} = 4\frac{\partial^2\psi}{\partial\overline{\mathbf{C}}\partial\overline{\mathbf{C}}}$, the components of $\mathbf{c}$ are given in terms of the Kroneker delta as,

$$c_{ijkl} = 2\mu \mathrm{III}^{-1}_{\overline{\mathbf{C}}}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}\right) \tag{3}$$

Above $\mathrm{III}_{\mathbf{c}}$ is the third principal invariant of $\overline{\mathbf{C}}$ or $\det\overline{\mathbf{C}}$. This material model is convenient for vulcanized and natural rubber. For other materials (e.g. Mooney-Rivlin) the elasticity tensor, see [3]. Now only the constitutive part namely the elasticity tensor $\mathbf{c}$ includes explicitly the topology design parameter μ. This linearized virtual work in the spatial description is often called *updated-Lagrangian* formulation. For further detail of nonlinear FEM, see [2],[3].

TOPOLOGY OPTIMIZATION

Problem formulation

We set the optimization problem such that only the end-compliance is minimized, so the FEM-discretized formula is as,

$$\min_{u,\rho_e} \mathbf{f}^T\mathbf{u}$$

$$s.t.: \left(\sum_{e=1}^{N} \rho_e^p \mathbf{K}^{(e)}\right)\mathbf{u} = \mathbf{f} \text{ or } \mathbf{R}(\mathbf{u}) = \left(\sum_{e=1}^{N} \rho_e^p \mathbf{K}^{(e)}\right)\mathbf{u} - \mathbf{f} = \mathbf{0}, \quad \sum_{e=1}^{N} \rho_e v_e \le V, \ 0 < \rho_{\min} \le \rho_e \le 1, e = 1, \dots, N. \tag{4}$$

I.e. the end-compliance is minimized as to the element's (e) material 'density' ρ_e under prescribed loading conditions subject to an upper bound on the effective volume. The elements in their lower bound are considered as void and others as solid. The penalization parameter p forces the densities to the lower or upper bound so that the optimized structure is so 'white and black' as possible. For incompressible material p is large, and to avoid the local minima or dependency of the resulting topology of the initial structure and optimization parameters, p should be slowly raised throughout the optimization (e.g. 1…>4). For further detail of setting the topology optimization problem, see [1]. $\mathbf{R}(\mathbf{u}) = \mathbf{0}$ represents the nonlinear governing equations of the model and $\mathbf{u}$ the vector of displacemets considered as state variables. The optimization problem above is solved using the interior penalty parameter or *augmented Lagrangian method* (ALM). The structural response is solved using the nonlinear quasi-static FEM based on N-R iteration and arc length method. Because of large deformations the line search method instead of straight application of N-R has to be used to improve convergence rate. The convergence criterion can be relaxed if oscillation occurs in nodes surrounded by void elements.

Sensitivity analysis

In gradient based optimization algorithms the sensitivities have to be determined for the objective and constraint functions. Assume the design independent loads and employ the adjoint method. In the equilibrium state for the objective the sensitivity is determined as follows:

$c(\rho,\mathbf{u}) = \mathbf{f}^T\mathbf{u}(\rho) - \lambda^T\mathbf{R}$ and $\dfrac{\partial c(\rho,\mathbf{u})}{\partial \rho_e} = \mathbf{f}^T\dfrac{d\mathbf{u}}{d\rho_e} - \lambda^T\left(\dfrac{\partial \mathbf{R}}{\partial \mathbf{u}}\dfrac{\partial \mathbf{u}}{\partial \rho_e} + \dfrac{\partial \mathbf{R}}{\partial \rho_e}\right)$. By requiring that all values of $\mathrm{d}\mathbf{u}/\mathrm{d}\rho_e$ must be satisfied: $\left(\mathbf{f}^T - \lambda^T\dfrac{\partial \mathbf{R}}{\partial \mathbf{u}}\right)\dfrac{\partial \mathbf{u}}{\partial \rho_e} = 0$, the Lagrangian multipliers λ, can be solved from the system $\left(\mathbf{f}^T - \lambda^T\dfrac{\partial \mathbf{R}}{\partial \mathbf{u}}\right) = \mathbf{0}$ or $\mathbf{K}_T\lambda = \mathbf{f}$. This provides an explicit expression for the objective function sensitivities (noting (2) and (3)):

$$\frac{\partial c}{\partial \rho_e} = -\lambda^T\frac{\partial \mathbf{R}}{\partial \rho_e} = -\lambda^T\frac{\partial \mathbf{K}_T}{\partial \rho_e}\mathbf{u} = -p\lambda^T\frac{\mathbf{K}_c^{(e)}}{\rho_e}\mathbf{u}. \tag{5}$$

The sensitivity of volume constraint is straightforward because of incompressibility. Filtering the sensitivities are used to ensure mesh-independency and to checkerboardcontrol. The filter is general used heuristic method based on modifying the design sensitivity of a spesific element as a weighted average over the element itself and its neighbours.

Numerical example: topology design for the slender plate

The aim is to determine the optimal distribution of the homogenous incompressible neo-Hookean material with μ equals to 600 kPa. Solving the structural response the simple four-noded bilinear quadrilateral elements are used. The plate is clamped on both sides, and three concentrated forces are acted on its top edge with values 1, 2 and 1 N. The dimension of the plate is 520×130×10 mm^3. Suppose 30 % of design domain volume material is available. The design topology using linear analysis is shown in Fig. 1(a) and from nonlinear analysis in Fig. 1(b). The optimum topology is such like that the exhibit snap-through behaviour may be occur [4], but not in this example, Fig. 1(c).

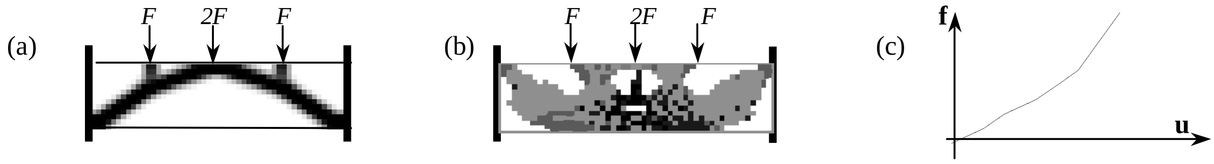

Figure 1. Compliance optimization problem. (a) The design domain, loads, boundary conditions and optimal topology using linear analysis. (b) Optimal topology from nonlinear analysis. (c) Load-disp.-curve of the optimum structure.

CONCLUSIONS

In this paper the topology optimization problem of fully nonlinear structure is studied. The mean compliance is chosen as an objective and the sensitivity analysis is derived using the adjoint method which is rather straightforward despite specially the material nonlinearity. The problem is solved iteratively by a sequential convex approximation method. Numerical examples indicate that for fully nonlinear structures the optimal result differs relatively much from the optimum of linear equivalent ones. For imcompressible nonlinear material achievement of the 'black-and-white' structure is hard and the convergence rate is slow . So the robust optimization solution methods should be used.

References

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