A UNIFIED TREATMENT FOR THE ELASTOPLASTIC BIFURCATION OF STRUCTURAL ELEMENTS

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Summary This work is devoted to a unified method for the analysis of the elastoplastic bifurcation and post-bifurcation of structural elements such as beams, plates and shells. In each case, the same bifurcation equation is solved, giving rise to analytical relations for the critical load, the eigenmode as well as the initial slope of the bifurcating branch which is essential for the stability analysis.

THE FRAMEWORK

The bifurcation analysis is carried out using the three-dimensional total Lagrangian formulation, where the Green strain is decomposed additively and the elastic strains are assumed small so that the constitutive law for the elastic behavior can be represented by the Saint-Venant-Kirchhoff relation. The plasticity is described within the frame of generalized standard materials, obeying the von Mises yield criterion and a linear isotropic hardening.
It is assumed that a fundamental equilibrium path $\lambda \mapsto \vec{u}_f(\lambda)$ is known, which is the displacement solution of the elastoplastic problem under increasing load λ. Moreover, at a critical instant t_c, it is assumed that there exists a bifurcating solution denoted $\vec{u}$, which is described by the asymptotic expansion :

$$\lambda = \lambda_c + \lambda_m \xi + o(\xi) \qquad\qquad \vec{u} = \vec{u}_f(\lambda) + \xi \vec{X} + o(\xi)$$

where the perturbation parameter ξ corresponds to the projection of the solution onto the buckling eigenmode.
Assuming the so-called tangent modulus hypothesis and uniqueness of the eigenmode, the critical load λ_c and the eigenmode $\vec{X}$ are solutions of the bifurcation equation [1],[2] :

$$\forall\ \delta\vec{u}, \qquad \int_\Omega \nabla^T \delta\vec{u} : \mathbf{K}^i(\vec{u}_f(\lambda_c)) : \nabla\vec{X} d\Omega = 0$$

where Ω is the region occupied by the body, $\mathbf{K}^i$ is either one of the following expressions, depending on the point belongs to the elastic or plastic region :

$$\mathbf{K}^e = \mathbf{F}.\mathbf{D}.\mathbf{F}^T + (\mathbf{I}.\Sigma)^T \qquad \text{or} \qquad \mathbf{K}^p = \mathbf{K}^e - \mathbf{M}^T \otimes \mathbf{M}, \quad \mathbf{M} = \frac{\mathbf{D} : \frac{\partial f}{\partial \Sigma}}{\sqrt{h + \frac{\partial f}{\partial \Sigma} : \mathbf{D} : \frac{\partial f}{\partial \Sigma}}}.\mathbf{F}^T$$

In the above, **F** denotes the gradient of the deformation, **D** the elasticity tensor, Σ the second Piola-Kirchhoff stress tensor (symmetric), **I** the fourth-order unit tensor, f the yield function and h the hardening modulus.
In the case where the elastic unloading zone at bifurcation is reduced to a single point, the initial slope λ_m of the bifurcating branch is given by :

$$\lambda_m = \min\{\lambda \mid \forall\, x \in \Omega,\ \lambda \mathbf{M}_c : \nabla\vec{u}_f,_\lambda(\lambda_c) + \mathbf{M}_c : \nabla\vec{X} \geq 0\} = \max_{x\in\Omega}\left(-\frac{\mathbf{M}_c : \nabla\vec{X}}{\mathbf{M}_c : \nabla\vec{u}_f,_\lambda(\lambda_c)}\right)$$

TIMOSHENKO BEAM UNDER AXIAL COMPRESSION

Throughout the sequel, we assume that the stress state in the body is uniaxial, directed in the x axis, and that the yield stress σ_0 is small enough for the buckling to occur when the body is wholly plastic.
Consider a straight cantilever beam with length L, constant cross-section area S, area moment of inertia I and tangent modulus E_T. The beam is subjected to a compressive load λ and its kinematics is described by the Timoshenko model. The fundamental solution writes, when the beam is wholly plastic ($\lambda \geq \sigma_0 S$) :

$$\vec{u}_f = -\frac{x}{S}\left(\frac{\lambda - \sigma_0 S}{E_T} + \frac{\sigma_0 S}{E}\right)\vec{x}$$

Assuming small pre-critical strains, the gradient tensor **F** can be replaced with the identity tensor. The components ($\mathcal{U}$,$\mathcal{V}$,Θ) defining the eigenmode are obtained from the bifurcation equation which simplifies as follows : $\forall\ \delta U, \delta V, \delta\theta$,

$$\int_\Omega \left[E_T(\mathcal{U},_x - y\Theta,_x)(\delta U,_x - y\delta\theta,_x) + \mu(\Theta - \mathcal{V},_x)(\delta\theta - \delta V,_x) - \frac{\lambda_c}{S}(\mathcal{U},_x - y\Theta,_x)(\delta U,_x - y\delta\theta,_x) - \frac{\lambda_c}{S}\mathcal{V},_x \delta V,_x\right] d\Omega = 0$$

It yields three scalar equations :

$$E_T S \mathcal{U},_x = 0 \qquad \mu S \Theta - \mu S \mathcal{V},_x + \lambda_c \mathcal{V},_x = 0 \qquad E_T I \Theta,_{xx} + \mu S \mathcal{V},_x - \mu S \Theta = 0$$

Taking into account the boundary conditions gives the critical load and the eigenmode :

$$\lambda_c = \frac{\frac{\pi^2 E_T I}{4L^2}}{1 + \frac{\pi^2 E_T I}{4L^2 \mu S}} \qquad \mathcal{U} = 0 \qquad \mathcal{V} = \left(1 + \frac{\pi^2 E_T I}{4L^2 \mu S}\right) \frac{2L}{\pi} \left(1 - \cos \frac{\pi x}{2L}\right) \qquad \Theta = \sin \frac{\pi x}{2L}$$

The well-known critical load for a Bernoulli beam is readily obtained from the above result. Moreover, one can get the initial slope of the bifurcating branch : $\lambda_m = 3\lambda_c$.

PLATE UNDER UNIAXIAL COMPRESSION

Consider a rectangular plate with dimensions a along $\vec{x}$, b along $\vec{y}$ ($a > b$ or $a < b$) and thickness h. The plate is subjected to a compressive stress Σ normal to sides of length b and its kinematics is described by the Love-Kirchhoff model. Denoting $\lambda = -\Sigma > 0$, the fundamental solution writes :

$$\vec{u}_f = -\left(\frac{\lambda}{E} + \frac{\lambda - \sigma_0}{h}\right) x\vec{x} + \left(\frac{\nu\lambda}{E} + \frac{\lambda - \sigma_0}{2h}\right) y\vec{y}$$

Solving the bifurcation equation shows that the out-of-plane component $\mathcal{W}$ of the eigenmode satisfies :

$$t^2(\alpha \mathcal{W},_{xxxx} + \gamma \mathcal{W},_{yyyy} + (2\beta + 4\mu)\mathcal{W},_{xxyy}) + 12\lambda_c \mathcal{W},_{xx} = 0$$

$$\text{where} \qquad \alpha = \frac{2\mu}{D}(E + 4h) \quad \beta = \frac{4\mu}{D}(E + 2\nu h) \quad \gamma = \frac{8\mu}{D}(E + h) \quad \text{with } D = 2\mu(5 - 4\nu) + 4h(1 - \nu)$$

By assuming a sinusoidal mode along $\vec{x}$ and $\vec{y}$, one gets the critical load and the eigenmode :

$$\lambda_c = \frac{t^2\pi^2}{12}\left(\frac{p^2\alpha}{a^2} + \frac{q^4\gamma a^2}{p^2 b^4} + \frac{q^2(2\beta + 4\mu)}{b^2}\right) \qquad \mathcal{U} = 0 \qquad \mathcal{V} = 0 \qquad \mathcal{W} = \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b}$$

The wave numbers corresponding to the smallest critical load are $p = \sqrt[4]{\frac{\gamma}{\alpha}}\frac{a}{b}$ along the load direction and $q = 1$. In particular, when $a = b = 1$ the above result leads to the value given in [3] : $\lambda_c = \frac{Et^2\pi^2}{12}\left(\frac{2}{1+\nu} + \frac{9 + \frac{E_T}{E}(8\nu - 1)}{(5-4\nu) - \frac{E_T}{E}(1-2\nu)^2}\right)$.

As in the beam case, one can also obtain the initial slope of the bifurcating branch : $\lambda_m = \frac{t\pi^2\left(\frac{2p^2}{a^2} - \frac{q^2}{b^2}\right)EE_T}{5E + E_T(2\nu - 1)}$.

TUBE UNDER AXIAL COMPRESSION

Let us now consider a circular tube with length l along axis $\vec{x}$, radius R and thickness t, subjected to a compressive axial stress Σ. Denoting $\lambda = -\Sigma > 0$, the fundamental solution reads in the cylindrical basis :

$$\vec{u}_f = -\left(\frac{\lambda}{E} + \frac{\lambda - \sigma_0}{h}\right) x\vec{x} + R\left(\frac{\nu\lambda}{E} + \frac{\lambda - \sigma_0}{2h}\right) \vec{e}_r$$

Solving the bifurcation equation in the axisymmetric case yields the coupled equations :

$$\alpha t \mathcal{U},_x + \frac{\beta}{R} t\mathcal{W} = 0 \qquad \alpha\frac{t^3}{12}\mathcal{W},_{xxxx} + \lambda_c t \mathcal{W},_{xx} + \frac{\gamma}{R^2} t\mathcal{W} + \frac{\beta}{R} t\mathcal{U},_x = 0$$

In the case of simply supported edges, the critical load can be computed by considering a sinusoidal eigenmode. After optimizing the wave number, one obtains $\lambda_c = \frac{Et}{R}\sqrt{\frac{4h}{3[(5-4\nu)E + 4h(1-\nu^2)]}}$ as well as the eigenmode :

$$\mathcal{U} = \frac{l}{n\pi R}\frac{2(E + 2\nu h)}{E + 4h}\left(\cos\frac{n\pi x}{l} - 1\right) \qquad \mathcal{V} = 0 \qquad \mathcal{W} = \sin\frac{n\pi x}{l} \quad \text{with } n = \frac{l}{\pi}\sqrt[4]{\frac{48h[(5 - 4\nu)E + 4h(1 - \nu^2)]}{R^2 t^2 (E + 4h)^2}}$$

The initial slope of the bifurcating branch is $\lambda_m = \frac{2Eh[5E + 4h + 8\nu h + \sqrt{48h[(5-4\nu)E + 4h(1-\nu^2)]}]}{R(E+4h)(5E+4h+2\nu h)}$.

References

[1] Hutchinson J.W.: Plastic buckling. *Advances in Applied Mechanics* **14**:67–144, 1974.
[2] Nguyen Q.S.: Stability and Nonlinear Solid Mechanics, Wiley, 2000.
[3] Cimetière, A.: Sur la modélisation et le flambage des plaques élastoplastiques. *PhD Thesis*, Université de Poitiers, France, 1987.