FRICTIONAL CONTACT WITH WEAR DIFFUSION

Meir Shillor*, Mircea Sofonea†, J. Joachim Telega‡
* *Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, USA*
† *LTS, Université de Perpignan, 52 Avenue de Villeneuve, 66860 Perpignan, France*
‡ *Polish Academy of Sciences IPPT PAN, Swietokrzyska 21, 00–049 Warsaw, Poland*

Summary A model for quasistatic frictional contact between a viscoelastic body and a moving foundation is described. The contact is modeled by the normal compliance and dry friction conditions and the wear of the contact surface is included. The novelty lies in allowing for the diffusion of the wear debris on the contact surface. Such phenomena arise in many settings, in particular in orthopedic biomechanics where they degrade the effectiveness of the joint prosthesis. A weak formulation of the problem is derived and, under a smallness assumption on the problem data, the existence of the unique weak solution for the model is stated.

THE MODEL

We model and analyze the processes when a viscoelastic body, which is acted upon by volume forces and surface tractions, is in frictional contact with a moving foundation and as a result a part of its surface may undergo wear. The wear particles or debris diffuse on the whole of the contact surface. Such situations arise, among others, in orthopedic biomechanics in the context of bone-implant interface. Since friction and the wear debris influence the quality and long term performance of artificial joints and implants, they need to be taken into account when modelling these processes, see, e.g., [1,2] and the references therein.

To describe the setting we denote by $\Omega \subset \mathbb{R}^3$ the domain occupied by the body and by Γ the boundary of Ω, which is assumed to be Lipschitz and is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C, such that $meas\,\Gamma_D > 0$. The body is clamped on Γ_D, prescribed surface tractions of density $\boldsymbol{f}_N$ act on Γ_N and volume forces of density $\boldsymbol{f}_0$ act in Ω. An initial gap g (≥ 0) exists between the potential contact surface Γ_C and the foundation, and is measured along the outward normal $\boldsymbol{\nu}$. To simplify the model we assume that the coordinate system is such that Γ_C occupies a regular domain in the plane $x_3 = 0$, the foundation is planar, and is moving with velocity $\boldsymbol{v}^*$ in the plane $x_3 = -g$. Furthermore, Γ_C is divided into two subdomains D_d and D_w by a smooth curve γ^*, and wear takes place only on the part D_w, while the diffusion of the wear particles takes place on the whole of Γ_C. The boundary $\gamma = \partial\Gamma_C$ of Γ_C is assumed Lipschitz and is composed of two parts γ_d and γ_w. Thus, $\partial D_w = \gamma_w \cup \gamma^*$ and $\partial D_d = \gamma_d \cup \gamma^*$. The setting is depicted in Fig 1.

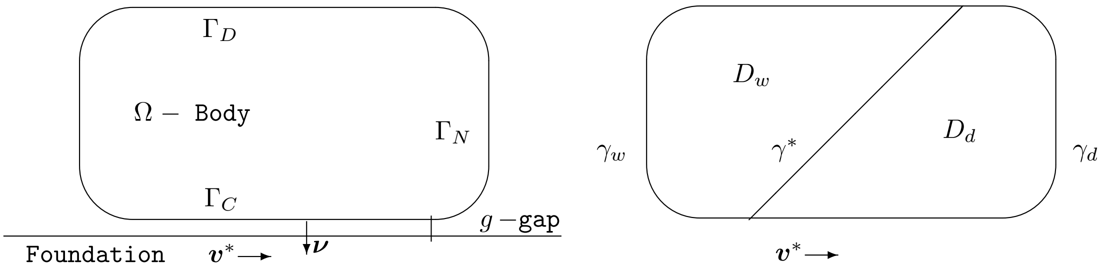

Fig. 1. The setting (left); contact surface Γ_C ; wear is produced in D_w (right).

We let $\mathbb{S}^3$ represent the space of second order symmetric tensors on $\mathbb{R}^3$ while " $\cdot$ " and $\|\cdot\|$ denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, respectively. We let $\boldsymbol{u}$ be the displacement vector, $\boldsymbol{\sigma}$ the stress field and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the linearized strain tensor. Below u_ν and $\boldsymbol{u}_\tau$ represent the *normal* and *tangential* displacements, and σ_ν and $\boldsymbol{\sigma}_\tau$ represent the *normal* and *tangential* stresses, respectively, Also $[0,T]$ denotes the time interval of interest, for $T > 0$, and a dot above a variable denotes the time derivative.

We describe the wear of the surface in terms of the *wear function* $w = w(\boldsymbol{x}, t)$ which is defined on D_w, and the diffusion of the wear particles by the *wear particle surface density function* $\zeta = \zeta(\boldsymbol{x}, t)$ which is defined on Γ_C. Here, $\boldsymbol{x} = (x_1, x_2, 0)$, since Γ_C lies in the plane Ox_1x_2. The wear function w measures the volume density of material removed per unit surface area (see, e.g., [4] and references therein).

The classical formulation of the problem of *frictional contact of a viscoelastic body with wear diffusion* is as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow \mathbb{R}^3$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \longrightarrow \mathbb{S}^3$*, the wear function* $w : D_w \times [0,T] \longrightarrow \mathbb{R}$ *and a surface particle density field* $\zeta : \Gamma_C \times [0,T] \longrightarrow \mathbb{R}$*, such that*

$$\boldsymbol{\sigma} = \mathcal{A}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \qquad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\text{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega \times (0,T), \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_D \times (0,T), \tag{3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \qquad \text{on } \Gamma_N \times (0,T), \tag{4}$$

$$-\sigma_\nu = p_\nu(u_\nu - w\chi_{[D_w]} - g) \qquad \text{on} \Gamma_C \times (0,T), \tag{5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le \mu|\sigma_\nu|; \qquad \boldsymbol{\sigma}_\tau = -\mu|\sigma_\nu|\frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{\|\dot{u}_\tau - \boldsymbol{v}^*\|} \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} \qquad \text{on } \Gamma_C \times (0,T), \tag{6}$$

$$\dot{w} = \mu p_\nu R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|) \qquad \text{on } D_w \times (0,T), \tag{7}$$

$$\dot{\zeta} - \operatorname{div}(k\,\nabla\zeta) = \kappa\mu p_\nu R^*(\|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)\chi_{[D_w]} \qquad \text{on } \Gamma_C \times (0,T), \tag{8}$$

$$\zeta = 0 \qquad \text{on} \quad \gamma \times (0,T), \tag{9}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \zeta(0) = \zeta_0 \qquad w(0) = w_0. \tag{10}$$

Here, (1) is the viscoelastic constitutive law of the material in which $\mathcal{A}$ and $\mathcal{G}$ are given nonlinear constitutive functions that may be nonhomogenuous; (2) are the equations of equilibrium, since the process is assumed quasistatic; (3) and (4) are the displacement and traction boundary conditions; (5) is the normal compliance contact condition, and (6) is the friction condition. The coefficient of friction $\mu = \mu(\zeta, \|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\|)$ is assumed to depend on the density of the wear particles and on the slip rate. Equation (7) is the pointwise Archard type wear production rate in D_w, while (8) is the diffusion equation for the wear particles in which k denotes the wear particle diffusion coefficient, κ is the wear rate coefficient, and $R^* : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator: $R^*(r) = r$ if $r \le R$, $R^*(r) = R$ if $r > R$, R being a given positive constant. This operator is needed to avoid some mathematical difficulties, but from the applied point of view the use of R^* is not restrictive since, the slip velocity is bounded. We use $\chi_{[D_w]}$ on the right-hand side of (8) since the particles are produced only in D_w, and the rate of production is multiplied by β. An absorbing boundary condition (9) is used, since once a wear particle reaches the boundary $\gamma = \partial\Gamma_C$ it is removed from the system. Finally, the initial conditions are given in (10) in which $\boldsymbol{u}_0, \zeta_0$ and w_0 are prescribed.

A version of problem (1)–(10) in which it was assumed that $\zeta = \beta w$ in D_w, where β is a conversion factor from wear depth to wear particles surface density, was studied in [3]. This simplification allows to eliminate the wear function w. However, here we present the full problem, since the wear w may be nonzero even when most of the wear debris is gone so that ζ is virtually zero.

MAIN RESULTS

In [3] we established the existence of a weak solution for the simplified model. The assumptions on the problem data were provided, and by using Green's formula we obtained a variational formulation for the problem. It is in the form of a system coupling an evolutionary variational inequality for the displacement field $\boldsymbol{u}$, an ordinary differential equation for w and a parabolic evolution equation for the surface particle density field ζ. Under a smallness assumption on the data the existence was proved. The proof was carried out in several steps, by using arguments of evolutionary equations, time-dependent elliptic variational inequalities and fixed point. The full details can be found in [3].

Here we report on our progress in dealing with the full problem (1)–(10).

CONCLUSIONS

The mathematical problem is new and important in applications. These results extend some of our results for contact problems to include wear diffusion. The relaxing of the assumption that w and ζ are proportional has mathematical and applied merit. The topic is currently under investigation and we hope to deal with the mathematical and numerical aspects of various variants of the model in the very near future.

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