EXISTENCE AND UNIQUENESS OF STEADY STATE SOLUTIONS IN THERMOELASTIC CONTACT WITH FRICTIONAL HEATING

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INTRODUCTION

It is well known that contact and friction in thermoelasticity result in mathematical problems which may lack solutions or may have multiple solutions. Previously, issues related to thermal contact and issues related to frictional heating have been discussed separately. In this work the two effects are coupled. Theorems of existence and uniqueness of solutions in two or three space dimensions are obtained, essentially extending, to frictional heating, results due to Duvaut, which were built on Barber's heat exchange conditions. Two qualitatively different existence results are given. The first one requires that the contact thermal resistance goes to zero at least as fast as the inverse of the contact pressure. The second existence theorem requires no such growth condition, but requires instead that the frictional heating, i.e., the sliding velocity times the friction coefficient, is small enough. Finally, it is shown that a solution is unique if the inverse of the contact thermal resistance is Lipschitz continues and the Lipschitz constant, as well as the frictional heating, is small enough.

VARIATIONAL FORMULATION

A thermoelastic body occupies a region Ω in R^d, $(d = 2, 3)$. The boundary of Ω contains three disjoint parts: Γ_0, Γ_1 and Γ_2. On Γ_0 the displacement vector u and the temperature T are prescribed to be zero. On Γ_1 the traction vector t and the heat flow q are prescribed. In the interior of Ω we have prescribed volume forces f and volume heat sources Q. Moreover, Γ_2 is the potential contact surface.
Now let us introduce function spaces V_1 and V_2 of displacements and temperatures, given by

$$V_1 = \{u \in (H^1(\Omega))^d : u = 0 \text{ on } \Gamma_0\},$$

$$V_2 = \{T \in H^1(\Omega) : T = 0 \text{ on } \Gamma_0\}$$

and the closed, convex subset

$$K_1 = \{u \in V_1 : u_N \le 0 \text{ on } \Gamma_2\}.$$

then, using Green's formula in the usual way, we may reformulate the classical equations for equilibrium of forces and heat production into the following coupled variational problems
Find $(u, T) \in K_1 \times V_2$ such that

$$a_1(u, v-u) - \int_\Omega T s_{ij} \frac{\partial(v_i - u_i)}{\partial x_j}\, dx \ge \langle L_1, v-u\rangle \tag{1}$$

$$a_2(T, \varphi) + \int_{\partial\Omega} \left[k(p^*)(T - T_0) - \mu V p^*\right] \varphi\, dS = \langle L_2, \varphi\rangle \tag{2}$$

for all $(v, \varphi) \in K_1 \times V_2$.
Here $a_1(\cdot,\cdot)$ is the bilinear form of elastic energy, the second term in (1) represents the thermoelastic coupling and the linear functional L_1 in the right hand side represents the external volume and traction forces.
In equation (2), $a_2(\cdot,\cdot)$ is the bilinear form of heat flow. The contact pressure is $p = -\sigma_{ij} n_i n_j$, and since this may only be expected to be a positive measure in the function space $H^{-1/2}(\partial\Omega)$ we have introduced a regularization operator

$$H^{-1/2}(\partial\Omega) \ni p \mapsto p^* \in L^2(\partial\Omega) \quad \text{which is linear and bounded} \tag{3}$$

with norm C^* so that

$$\|p^*\|_{L^2(\partial\Omega)} \le C^* \|p\|_{-1/2,\partial\Omega} \tag{4}$$

and that

$$p \ge 0 \Longrightarrow p^* \ge 0. \tag{5}$$

In addition we require that if p is a measure on $\partial\Omega$ then

$$p \ge 0 \Longrightarrow \|p^*\|_{L^\infty(\partial\Omega)} \le c_0 \|p\|_{\mathcal{M}(\partial\Omega)} \tag{6}$$

where c_0 is independent of p and $\|\cdot\|_{\mathcal{M}(\partial\Omega)}$ denotes the total variation-norm on the space $\mathcal{M}(\partial\Omega)$ of bounded measures on $\partial\Omega$. The conditions (3)-(6) are certainly satisfied if for example the mapping * is given by a convolution (averaging) with a non-negative, piecewise C^1-function having compact support.
The heat balance of the contact interface is now represented by the second term in (2). Frictional heating produced on the boundary of the body, where μ is coefficient of friction and V is a given tangential velocity of the obstacle. The sum of these two heat flow contributions leaves into a perfectly conducting obstacle that has the temperature T_0. However, it then has to pass through a thermal resistance

$$R = 1/k(p^*),$$

where $k(p)$ is non-negative. The right hand side with the linear functional L_1 represents the external heat sources.

TWO THEOREMS OF EXISTENCE

In this section we will formulate two theorems of existence for the frictional thermoelastic problem. The first theorem deals with the case when the thermal conductance $k(p)$ is at least linearly increasing as the contact pressure p tends to infinity.

Theorem 1 *If $k(p) = 0$ for $p \leq 0$, if $mp \leq k(p)$ with $m > 0$ and if k is continuous, then the problem defined by (1) and (2) has at least one solution (u, T). Moreover, for this solution we have*

$$\|T\|_{H^1(\Omega)} \leq C(V\|\mu\|_{1/2,\partial\Omega}, 1/m)$$

with $C(\cdot,\cdot)$ an increasing function of both arguments.

The idea of the growth condition in this theorem came from the fact that if equality is assumed, i.e., $mp = k(p)$, one may define a modified temperature $T_1 = T_0 + \mu V/m \in H^1(\Omega)$, such that the integrand of the integral reads $k(p^*)(T - T_1)$. This means that our problem takes the form already considered by Duvaut (1979) for the case of no frictional heating and an existence result follows by analogy.
In the second theorem no growth condition is required for k. Instead there is a restriction on the size of μV *i.e.*, on the coefficient of friction and the velocity. On the other hand the regularity of μ is less restricted, only $\mu \in L^2(\partial\Omega)$ or $\mu \in L^\infty(\partial\Omega)$ is needed.

Theorem 2 *If $k(p) = 0$ for $p \leq 0$, if $k(p) \geq 0$ for $p \geq 0$, if k is continuous and if*

$$V\|\mu\|_{L^2(\partial\Omega)} < c_1 c_2 / c_0 c_4 C_1^2 \|S\| \|\mathcal{E}\|^2 \tag{7}$$

or

$$V\|\mu\|_{L^\infty(\partial\Omega)} < c_1 c_2 / C^* c_4 C_1 \|S\| \|\mathcal{E}\| \tag{8}$$

then the problem defined by by (1) and (2) has at least one solution (u, T).

Here the constants c_1, c_2, C_1, C_2 are related to the bilinear forms $a_1(\cdot,\cdot)$ and $a_1(\cdot,\cdot)$, c_4 to a trace theorem and $\|\mathcal{E}\|$ is the norm of an extension operator from $H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$.

UNIQUENESS

To prove a theorem on uniqueness of solutions we make the additional assumption that the function k is Lipschitz continuous.

Theorem 3 *The solutions proven to exist in Theorems 1 and 2 are unique if k is Lipschitz continuous with Lipschitz constant L, and if L and the sliding velocity V are small enough.*

References

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