# SENSITIVITY ANALYSIS CONCERNING THE INITIAL POSTBUCKLING BEHAVIOR OF ELASTIC STRUCTURES

Herbert A. Mang* and Christian Schranz*
*Institute for Strength of Materials, Vienna University of Technology, Karlsplatz 13, A 1040 Vienna, Austria

*Summary* Many engineering structures are imperfection-sensitive. A conversion to an imperfection-insensitive structure may be achieved by specific modes of stiffening. In this paper mathematical relations describing the sensitivity of the initial postbuckling behavior to such modes of stiffening is presented. Koiter's initial postbuckling analysis is applied in the context of the Finite Element Method to deduce these relations. They permit to determine whether or not a specific mode will result in a transition from imperfection sensitivity to insensitivity. Two relatively simple numerical examples serve as the vehicle to corroborate the theoretical findings.

## INTRODUCTION

In order to improve the mechanical behavior of imperfection-sensitive structures, it is not sufficient to understand the influence of a specific mode of stiffening of a structure on the prebuckling behavior and the stability limit. What is equally important is knowledge about the corresponding influence on the postbuckling response. This was recognized, e.g., by Bochenek and Krużelecki [1] who have dealt with the challenging research topic of optimization of the postbuckling behavior of elastic structures. In case of loss of stability by means of symmetric bifurcation, the improvement of the (initial) postbuckling response by suitable stiffening may result in a conversion from an originally imperfection-sensitive to an imperfection-insensitive structure. Designation of a structure as either imperfection-sensitive or insensitive depends on the initial postbuckling behavior [2] which may (but need not) be relevant for the entire postbuckling response. In case of symmetric bifurcation, the mechanical consequence of these characteristics is loss of stability of an imperfect structure by snap-through at a load level that may be significantly lower than the one of the first bifurcation point on the primary path of the perfect structure. Hence, a conversion from an originally imperfection-sensitive to an imperfection-insensitive structure is desirable. This has been the motivation for sensitivity analyses concerning the initial postbuckling behavior of elastic structures with respect to specific modes of stiffening.

Conclusions of the present work will be drawn from the sensitivities of the parameters $a_1$, $\lambda_2$, and $\lambda_4$, where $a_1$ is the so-called nonlinearity coefficient, which is trivially zero for linear prebuckling paths, and $\lambda_2$ and $\lambda_4$ are the first two (generally) non-vanishing parameters in the asymptotic expansion of a load increment in terms of a path parameter for the secondary paths [2]. These quantities depend on a design parameter $\kappa$ such as, e.g., the thickness of the structure or the stiffness of a suitably attached spring.

## THEORETICAL INVESTIGATION

In Mang et al [3] details of Koiter's initial postbuckling analysis [2] in the context of the FEM [4] can be found. It is employed to derive a relationship between the three aforementioned parameters – $\lambda_2$, $\lambda_4$, $a_1$ – playing a central role in case of symmetric bifurcation from nonlinear primary paths [3]. The parameter $\lambda_2$ is related to the initial slope, whereas the parameters $\lambda_2$ and $\lambda_4$ are related to the initial curvature of secondary paths at the bifurcation point. The third parameter, $a_1$, is closely related to the curvature of the so-called "eigenvalue curve" at the stability limit, based on the consistently linearized eigenvalue problem [5].

Using Koiter's initial postbuckling analysis in the context of the FEM, the following relations can be derived:

$$a_1\,\lambda_2^2 - d_3 + \lambda_4 = 0 \quad \text{and} \quad 2\,a_1\,\lambda_4 + b_4 = 0\,, \tag{1}$$

where $d_3$ and $b_4$ are additional parameters obtained from Koiter's initial postbuckling analysis. These two relations are the basis for assessing whether or not an increase of $\kappa$ at $\lambda_2 = 0$ results in a transition from imperfection sensitivity to insensitivity. It is shown that the case of interest, i.e., $\lambda_2 = 0$, is characterized either by $a_1 = 0$ or $1/a_1 = 0$ and by special mathematical properties of the eigenvalue curve and the corresponding eigenvector curve, both depending on the load parameter. The common feature of the eigenvector curves at $\lambda_2 = 0$ is the existence of a singular point of the vector function $\mathbf{u}_1^*(\lambda)$ at the stability limit, indicating a fundamental change of the mechanical behavior of the structure. In case of nonlinear prebuckling paths, $a_1 = 0$ is a necessary condition for the aforementioned transition. Performing a sensitivity analysis on Equation (1.1) and specializing the result for this case, i.e., for $a_1 = 0$ and $\lambda_2 = 0$, yields the following relation for the rates of change of $a_1$ and $\lambda_2$ with respect to $\kappa$:

$$a_{1,\kappa}\;\lambda_{2,\kappa} = 0\,. \tag{2}$$

Six different sensitivity modes containing the combination of interest, i.e., $\lambda_2 = 0$, $a_1 = 0$, are shown to exist for structures with nonlinear prebuckling paths. However, only three of them lead to a transition from imperfection sensitivity to imperfection insensitivity. These modes are presented in form of $\lambda_2$-$\lambda_4$-$a_1$ diagrams which reflect the change of these parameters in the course of varying the design parameter $\kappa$, e.g., the stiffness of a suitably attached elastic spring, the thickness of the structure, or the rise of the undeformed structure.

Figure 1 shows $\lambda_2$-$\lambda_4$-$a_1$ diagrams of two of the six modes. The arrows on the curves indicate the direction of increase of the design parameter $\kappa$. The starting point is denoted as $S$ and the final point as $F$. Point $T$ is characterized by $\lambda_2 = 0$ and, hence, indicates a candidate point for a transition from imperfection sensitivity to insensitivity. Point $D$, characterized by $a_1 = -\infty$, indicates loss of stability by means of snap-through. Figure 1(a) illustrates a mode of stiffening for which a transition from imperfection sensitivity ($\lambda_2 < 0$) to insensitivity ($\lambda_2 > 0$) takes place. In contrast, the mode of stiffening illustrated in Figure 1(b) reaches point $T$, but does not lead to a transition to imperfection insensitivity.

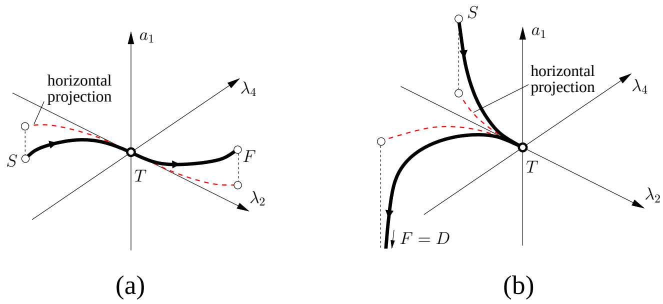


Figure 1: Qualitative plots of curves $\lambda_2 = \lambda_2(\kappa)$, $\lambda_4 = \lambda_4(\kappa)$, $a_1 = a_1(\kappa)$, with one point $T(\lambda_2 = 0, \lambda_4 = 0, a_1 = 0)$ each

## NUMERICAL INVESTIGATION

The numerical investigation [6] involves two different structures which are both imperfection-sensitive. Hence, $\lambda_2$, which is related to the initial slope of the postbuckling path [3], is negative.

The first structure is a *von Mises* truss. As a first attempt to improve the initial postbuckling behavior, a spring is applied at the load point. Increasing the stiffness of the spring leads to an increase of $\lambda_2$. Eventually, the *von Mises* truss becomes imperfection-insensitive. The corresponding $\lambda_2$-$\lambda_4$-$a_1$ diagram is shown in Figure 1(a).

The second structure is a shallow cylindrical shell with a point load at the center. Attempting to improve the initial postbuckling behavior, the thickness of the shell is increased. At first, $\lambda_2$ increases and becomes zero for a specific value of the thickness (see point $T$ in Figure 1(b)). However, if the thickness is further increased, $\lambda_2$ becomes negative again, and the structure stays imperfection-sensitive.

Next, an elastic spring is attached at the load point for different thicknesses of the shell. Increasing the stiffness of a spring for a thick shell leads to a transition from imperfection sensitivity to insensitivity. Hence, $\lambda_2$ becomes positive. For a thin shell, however, $\lambda_2$ stays negative and reaches zero only for the asymptotic limiting value of an infinite spring stiffness. Hence, if the thickness of the shell is smaller than a problem-dependent limiting value, a transition from $\lambda_2 < 0$ to $\lambda_2 > 0$ and, consequently, a conversion from imperfection sensitivity to insensitivity, is impossible.

Choosing the rise of the undeformed structure as the design parameter $\kappa$, it depends on the initial configuration of the two structures whether the decrease of the rise will result in a transition to snap-through or to no loss of stability.

## CONCLUSIONS

Depending on the mode of stiffening, in case of symmetric bifurcation a transition from imperfection sensitivity to insensitivity is possible. Two simple structures were chosen as a vehicle for verification of the theoretical findings. Increasing the stiffness of a suitably attached spring leads to a conversion of the imperfection-sensitive *von Mises* truss to an imperfection-insensitive structure. In case of the shallow cylindrical shell, a minimum thickness is required for such a conversion to occur. Changing the rise of a structure has no influence on the parameter $\lambda_2$. Hence, it will not result in a transition from imperfection sensitivity to insensitivity. Increasing the stiffness through a uniform increase of the thickness of the structure does also not result in such a transition.

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