THE ABSOLUTE COORDINATE FORMULATION WITH REDUCED STRAIN AND STIFFENING

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Summary The present work contributes to the field of multibody systems with respect to the absolute coordinate formulation (ACF) with a reduced expression of the strain energy. The ACF is known to be advantageous with respect to the description of constraint equations and the constant mass matrix. Absolute coordinates are used as unknowns, similar to nonlinear finite element methods. In the present work, a consistent linearization of the equations of motion with respect to small deformations but large rigid-body motions is performed. This formulation leads to a constant mass matrix while the nonlinear stiffness matrix is composed of the constant small strain stiffness matrix rotated by the underlying rigid body rotation. The equations of motion are derived for the case of a constrained multibody system and geometrical stiffening terms are introduced.

INTRODUCTION

The computation of stress and vibration in heavily loaded machine elements is necessary in order to design such parts for a specific lifetime. Since the beginning of multibody system dynamics, several computational strategies emerged, see e.g. Shabana[6]. The methodologies can be separated into ones that use absolute coordinates and a nonlinear strain-displacement relation and others that use relative coordinates with respect to a co-moving frame and a linear strain-displacement relation, also known as floating frame of reference formulation (FFRF), which leads to a constant stiffness matrix but highly nonlinear mass matrix. Absolute coordinates, which can be e.g. displacements or displacement gradients, have been utilized according to standard finite element formulations or specific beam, plate or shell models have been derived, see e.g. Mikkola and Shabana[5]. Recently, the ACF, which is basically equivalent to a geometrically nonlinear finite element formulation with specific generalized coordinates, has been extended with respect to a reduced strain formulation, see Gerstmayr[3]. The idea of the reduced strain formulation is to linearize the geometrically nonlinear Green strain tensor with respect to a virtual co-moving frame similar to the FFRF. The linearization is based on the rotation of the co-moving frame and leads to a constant mass matrix, but nonlinear stiffness matrix. The method has already been tested with for 3D solid multibody systems with constraints, see Gerstmayr and Schöberl[3], and extended with respect to elasto-plasticity, see Gerstmayr[2]. The method turned out to be computationally very efficient in the case of implicit Runge Kutta time integration schemes, see Gerstmayr and Schöberl[4]. As a specific feature of the method, the mass matrix is constant, the stiffness matrix is consisting of the transformed (rotated) constant stiffness matrix and some additional small nonlinear terms.

ABSOLUTE COORDINATE FORMULATION WITH A REDUCED STRAIN TENSOR

The derivation of the equations of motion starts with the weak form of the equations of motion without any simplifications regarding deformation or displacements,

$$\int_{V_0} \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dV_0 + \int_{V_0} \mathbf{S} : \delta\mathbf{E}\, dV_0 - \int_{V_0} \mathbf{f}_0 \cdot \delta\mathbf{u}\, dV_0 = 0 \tag{1}$$

The Lagrangian formulation is used throughout, $\mathbf{S}$ denotes the 2^{nd} Piola-Kirchhoff stress tensor and $\mathbf{E}$ is the Green strain tensor. External forces are denoted by $\mathbf{f}_0$, $\mathbf{u}$ are the unknown displacements, V_0 is the initial Volume and ρ is the material density. The goal of the present formulation is to neglect some terms in the virtual work of internal forces according to the widely used assumptions in multibody systems, such that the deformations are small and that displacements are small with respect to an underlying rigid body motion. According to Figure 1, the displacement of a point $\mathbf{X}$ is split into a rigid part $\mathbf{u}_r$ and a flexible part $\mathbf{u}_s$ defined by

$$\mathbf{u} = \mathbf{u}_0 + (\mathbf{R} - \mathbf{I})\mathbf{X} + \mathbf{u}_s = \mathbf{u}_r + \mathbf{u}_s. \tag{2}$$

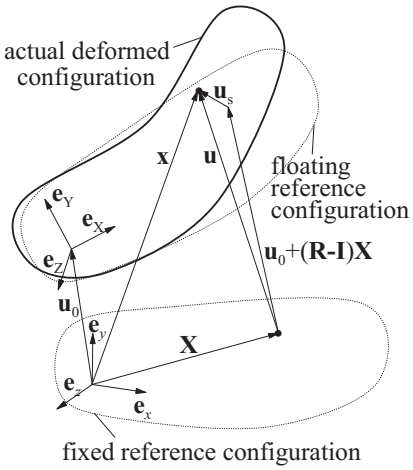

Figure 1: Kinematical relations, small deformation $\mathbf{u}_s$ and large rotation $\mathbf{R}$.

The unknowns are $\mathbf{u}$ and Eq. (2) is used to compute the flexible part $\mathbf{u}_s$ as function of $\mathbf{u}$. This kind of decomposition is a priori non-unique. The rotation matrix $\mathbf{R}$ and the displacement of the reference point $\mathbf{u}_0$ depend either on the whole displacement field $\{\mathbf{u}(\mathbf{X}), \mathbf{X} \in V_0\}$ or on some parts of it. Examplarily, we can use three specific points such as

$$\mathbf{R} = \mathbf{R}(\mathbf{u}(\mathbf{X}_1), \mathbf{u}(\mathbf{X}_2), \mathbf{u}(\mathbf{X}_3)) \quad \text{and} \quad \mathbf{u}_0 = \mathbf{u}_0(\mathbf{u}(\mathbf{X}_1)). \tag{3}$$

The dependence onto the three displacements might be highly nonlinear, and a certain orthogonalization rule has to be considered within this function. The coordinates $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ are chosen similar to the definition of the well known

chord or tangential frames in the FFRF. Alternatively, weighted integrals can be used for the definition of mean axis frames, see Gerstmayr and Schöberl[3]. The flexible part of displacements $\mathbf{u}_s$ can be written as function of $\mathbf{u}$

$$\mathbf{u}_s = \mathbf{u} - \mathbf{u}_0 - (\mathbf{R} - \mathbf{I})\mathbf{X}. \tag{4}$$

The fully nonlinear Green strain tensor is defined by

$$\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u}\right) = \frac{1}{2}\left(\nabla\mathbf{u}_s^T\mathbf{R} + \mathbf{R}^T\nabla\mathbf{u}_s + \nabla\mathbf{u}_s^T\nabla\mathbf{u}_s\right). \tag{5}$$

In the FFRF deformations are frequently assumed to be small, the linearized strain tensor is utilized and therefore the stiffness matrix is constant. It can be shown that this assumption is equivalent to $||(\nabla\mathbf{u}_s)|| \ll 1$ in the proposed ACF. Thus, a reduced Green strain tensor $\widetilde{\mathbf{E}}$ is introduced, where the quadratic part $\widetilde{\mathbf{E}}_{nl} = \frac{1}{2}\nabla\mathbf{u}_s^T\nabla\mathbf{u}_s$ is assumed to be negligible, $||\widetilde{\mathbf{E}}_{nl}|| \ll 1$,

$$\widetilde{\mathbf{E}} = \mathrm{Sym}(\mathbf{R}^T\nabla\mathbf{u}_s) = \mathrm{Sym}(\mathbf{R}^T(\nabla\mathbf{u} + \mathbf{I}) - \mathbf{I}). \tag{6}$$

The variation of this tensor is $\delta\widetilde{\mathbf{E}} = \mathrm{Sym}(\delta\mathbf{R}^T\nabla\mathbf{u}_s + \mathbf{R}^T\delta\nabla\mathbf{u}_s)$ and contains the cumbersome variation of the rotation matrix $\mathbf{R}$. The reduced Green strain tensor of Eq. (6) is inserted into Eq. (1). A weak formulation for $\mathbf{u}$ is derived by means of a linear elastic constitutive law, $\mathbf{S} = \lambda\mathrm{tr}(\widetilde{\mathbf{E}})\mathbf{I} + 2\mu\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}} : {}^4\mathbf{D}$, with the Lamé coefficients μ and λ and the identity matrix $\mathbf{I}$. According to Eq. (1), the ACF with the reduced Green strain tensor reads for the linear elastic case

$$\int_{V_0} \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dV_0 + \int_{V_0} \widetilde{\mathbf{E}} : {}^4\mathbf{D} : \delta\widetilde{\mathbf{E}}\, dV_0 = \int_{V_0} \mathbf{f}_0 \cdot \delta\mathbf{u}\, dV_0. \tag{7}$$

According to Zienkiewicz and Taylor[7], the geometrical stiffening could be represented by means of additional terms in the virtual work of internal forces, such as $\int_{V_0}\mathbf{S}_0 : \delta\widetilde{\mathbf{E}}_{nl}\, dV_0$, where $\mathbf{S}_0$ is represents an appropriate approximation of stresses within the body of interest. In multibody systems, it can be sufficient to approximate these terms by means of centrifugal forces due to high angular velocities of the underlying rigid body motion. The final weak form of equations of motion reads

$$\int_{V_0} \rho\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dV_0 + \int_{V_0} \widetilde{\mathbf{E}} : {}^4\mathbf{D} : \delta\widetilde{\mathbf{E}}\, dV_0 + \int_{V_0} \mathbf{S}_0 : \delta\widetilde{\mathbf{E}}_{nl}\, dV_0 = \int_{V_0} \mathbf{f}_0 \cdot \delta\mathbf{u}\, dV_0. \tag{8}$$

The discretized equations follow from a particular finite element discretization, see e.g. Zienkiewicz and Taylor[7]. According to standard constraint formulations in multibody system dynamics based on Lagrange multipliers we write

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + {}^h\mathbf{R}\mathbf{K}\,{}^h\mathbf{R}^T(\mathbf{q} - \mathbf{q}_r) + \mathbf{f}_{\mathrm{nl}}(\mathbf{q}) + \mathbf{f}_{\mathrm{stiff}}(\mathbf{q}) + \left(\frac{\partial\mathbf{C}}{\partial\mathbf{q}}\right)^T\lambda &= \mathbf{f}_{ext} \\ \mathbf{C}(\mathbf{q}) &= 0 \end{aligned} \tag{9}$$

The kinetic energy leads to a constant mass matrix where $\mathbf{q}$ are the discretized displacements, $\mathbf{q}_r$ is the rigid body part of discretized displacements, $\mathbf{M}$ is the constant mass matrix, $\mathbf{K}$ is the constant small strain stiffness matrix of the body in the reference configuration, $\mathbf{f}_{ext}$ are the external forces, $\mathbf{f}_{nl}$ is a nonlinear function of $\mathbf{q}$ which is of the order $\mathbf{q}_s^2$ and $\mathbf{f}_{\mathrm{stiff}}$ is the part due to stiffening. The matrix ${}^h\mathbf{R}$ consists of sub-matrices $\mathbf{R}$ in the diagonal. In previous investigations it turned out that Eq. (9) can be solved efficiently, see Gerstmayr and Schöberl[4]. Due to special properties of the mass and the stiffness and rotation matrices, only a nonlinear system of the size of the constraint forces λ has to be solved during each time step of an implicit time integration method, assuming the nonlinear parts to be solved in the fixed point iteration. In the case of small deformations, the nonlinear part and the part due to stiffening are computed by means of an additional fixed point iteration, which converges within a few iterations. Therefore, no large system of nonlinear equations has to be solved, and no large Jacobian or stiffness matrix of a large nonlinear system has to be assembled during every time step.

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