SIMULATION OF CONSTRAINED RIGID AND ELASTIC BODIES WITHOUT CONSTRAINT EQUATIONS

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Summary Mechanical systems of both rigid and elastic bodies are considered. Differential-algebraic equations (DAE) of their motion and a technique of elimination of the algebraic part to obtain ordinary differential equations (ODE) are discussed.

INTRODUCTION

The most common case of coupling rigid and elastic bodies 1 and 2 assumes employing DAE like these:

$$\begin{aligned} \mathbf{M}_1\ddot{\mathbf{x}}_1 &= \mathbf{f}_1 + (\mathbf{G}_1^{\mathrm{T}}\boldsymbol{\lambda}) \\ \mathbf{M}_2\ddot{\mathbf{x}}_2 &= \mathbf{f}_2 + (\mathbf{G}_2^{\mathrm{T}}\boldsymbol{\lambda}) \\ (\mathbf{g}(\mathbf{x}_1,\mathbf{x}_2) &= \mathbf{0}) \end{aligned} \qquad (1)$$

with mass matrices $\mathbf{M}_i$, generalized forces $\mathbf{f}_i$ and coordinates $\mathbf{x}_i$ of the two bodies. Constraints $\mathbf{g}$ result in reaction forces $\mathbf{G}_i^{\mathrm{T}}\boldsymbol{\lambda}$ with Jacobian matrices $\mathbf{G}_i = \partial\mathbf{g}/\partial\mathbf{x}_i^{\mathrm{T}}$ and undefined Lagrangian multipliers $\boldsymbol{\lambda}$: dim $\boldsymbol{\lambda}$ = dim $\mathbf{g}$.
DAE introduce additional difficulties into numerical investigations such as problems of constraint violations. They can be successfully solved using special methods, e.g. [5]. However in many cases it is possible to avoid DAE, for example, in the finite element method (FEM).

THE ASSEMBLING PROCEDURE IN FEM

Let us consider two finite elements shown as beams in Fig. 1. Let 1st FE's nodal coordinates are divided on two parts $\mathbf{x}_1$ and $\mathbf{x}_2$, while those for 2nd one are $\mathbf{x}_3$ and $\mathbf{x}_4$, so that $\mathbf{x}_2$ and $\mathbf{x}_3$ are compatible (the elements can be linked in these variables).

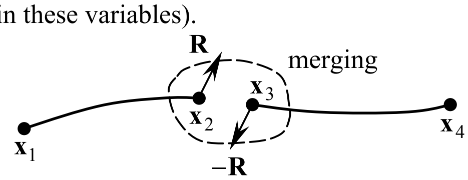

Figure 1. Assembling two finite elements

Equations of motion of the two elements are:

$$\begin{cases} \mathbf{M}_{11}\ddot{\mathbf{x}}_1 + \mathbf{M}_{12}\ddot{\mathbf{x}}_2 = \mathbf{f}_1 \\ \mathbf{M}_{21}\ddot{\mathbf{x}}_1 + \mathbf{M}_{22}\ddot{\mathbf{x}}_2 = \mathbf{f}_2 + (\mathbf{R}) \end{cases}$$

$$\begin{cases} \mathbf{M}_{33}\ddot{\mathbf{x}}_3 + \mathbf{M}_{34}\ddot{\mathbf{x}}_4 = \mathbf{f}_3 - (\mathbf{R}) \\ \mathbf{M}_{43}\ddot{\mathbf{x}}_3 + \mathbf{M}_{44}\ddot{\mathbf{x}}_4 = \mathbf{f}_4 \end{cases}$$

$$(\mathbf{x}_2 \equiv \mathbf{x}_3 = \mathbf{x}_*)$$

with mass matrices $\mathbf{M}_{ij}$, generalized forces $\mathbf{f}_i$ and reaction forces $\mathbf{R}$. Terms in brackets appear when the elements are joined as shown in Fig. 1. The equations have DAE form (1) but the constraints are trivial. Eliminating $\mathbf{R}$ from the second and third equations and accounting the fifth one leads to ODE

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{O} \\ \mathbf{M}_{21} & \mathbf{M}_{22} + \mathbf{M}_{33} & \mathbf{M}_{34} \\ \mathbf{O} & \mathbf{M}_{43} & \mathbf{M}_{44} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_1 \\ \ddot{\mathbf{x}}_* \\ \ddot{\mathbf{x}}_4 \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 + \mathbf{f}_3 \\ \mathbf{f}_4 \end{Bmatrix}$$

where the whole system mass matrix and generalized forces are composed of that for separated elements.
Further we will use this idea to assemble rigid and elastic bodies to each other.

Compatible coordinates of rigid and elastic bodies

The possibility of eliminating constraint equations depends on compatibility of coordinates of rigid and elastic bodies ($\mathbf{x}_2$ and $\mathbf{x}_3$ above). Rigid bodies usually have rotation angles as generalized coordinates: e.g. φ in 2D case or any triplet α, β, γ in 3D case.
We consider large displacement finite-element formulations for simulation of elastic bodies. One of them is the *large rotation vector formulation* [3], which employs absolute rotation angles. For example, plane beam elements in Fig. 1 have the following structure of generalized coordinates: $\mathbf{x}_i = \{x_i, y_i, \varphi_i\}^{\mathrm{T}}$, where x_i and y_i are absolute Cartesian coordinates of i-th node while φ_i is the absolute rotation angle of the beam cross section. 3D beam and plate elements in this formulation [2] have nodal coordinates like these: $\mathbf{x}_i = \{x_i, y_i, z_i, \alpha_i, \beta_i, \gamma_i\}^{\mathrm{T}}$. Obviously, these finite elements can be easily assembled with rigid bodies due to coordinate compatibility.

FEM formulations with incompatible coordinates

Another large displacement finite-element approach is the *absolute nodal coordinate formulation* (ANCF). The structure of nodal coordinates for beam elements in this method is as follows (cf. Fig. 1 and 2): $\mathbf{x}_i = \{\mathbf{r}_i^{\mathrm{T}}, \boldsymbol{\tau}_i^{\mathrm{T}}\}^{\mathrm{T}}$, where $\mathbf{r}_i$ and $\boldsymbol{\tau}_i$ are the absolute radius vectors and the slope vectors. This approach leads to a constant mass matrix and zero centrifugal and Coriolis inertia forces, in contrast to other large displacement formulations [6].
It is clear that attaching a 2D rigid body to such an

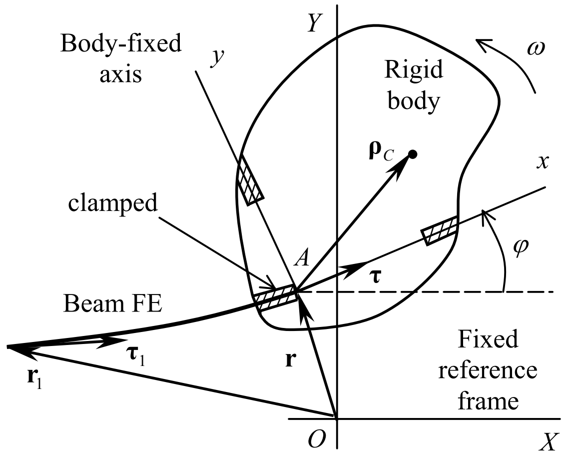

Figure 2. Rigid body attached to a beam element

ANCF beam element [7] leads to the constraint equation

$$\varphi = \arctan(\tau_Y/\tau_X) \quad (2)$$

and, therefore, to differential-algebraic equations (1).
To eliminate the constraints we must make the rigid body coordinates compatible to that for beam element:

$$\mathbf{x} = \{\mathbf{r}^{\mathrm{T}}, \boldsymbol{\tau}^{\mathrm{T}}\}^{\mathrm{T}}. \quad (3)$$

We introduce four scalar coordinates for planar rigid body and must redevelop its equations of motion [7].
Angular velocity of the body is derived from (2):

$$\omega = \dot{\varphi} = \frac{\dot{\boldsymbol{\tau}}^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\tau}}{\tau^2}, \text{ where } \tilde{\mathbf{I}} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}, \ \tau^2 = \tau_X^2 + \tau_Y^2.$$

Velocity vector of an arbitrary point $\boldsymbol{\rho}$ of the body is

$$\mathbf{v} = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\boldsymbol{\rho} = \boldsymbol{\Phi}\dot{\mathbf{x}},$$

where the velocity transformation Jacobian matrix $\boldsymbol{\Phi} = \begin{bmatrix}\mathbf{I} & \tilde{\mathbf{I}}\boldsymbol{\rho}\boldsymbol{\omega}_{\dot{\tau}}^{\mathrm{T}}\end{bmatrix}$ is introduced, $\boldsymbol{\omega}_{\dot{\tau}} = \partial\omega/\partial\dot{\boldsymbol{\tau}} = \tilde{\mathbf{I}}\boldsymbol{\tau}/\tau^2$.
Then, we use the principle of virtual work

$$\int_V \delta\mathbf{r}^{\mathrm{T}}\mu(\mathbf{a}-\mathbf{g})\,\mathrm{d}V = 0 \quad (4)$$

with virtual displacement $\delta\mathbf{r} = \boldsymbol{\Phi}\delta\mathbf{x}$ and acceleration $\mathbf{a} = \dot{\mathbf{v}} = \boldsymbol{\Phi}\ddot{\mathbf{x}} + \dot{\boldsymbol{\Phi}}\dot{\mathbf{x}}$ of a body point $\boldsymbol{\rho}$. Integration in the body volume V is assumed. μ is the body mass density while $\mathbf{g}$ is the gravity acceleration.
Equations of motion (4) take the matrix form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{f}^{\mathrm{inert}} = \mathbf{f}^{\mathrm{grav}}, \quad (5)$$

$$\mathbf{M}(\mathbf{x}) = \int_V \mu\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\,\mathrm{d}V = \begin{bmatrix} m\mathbf{I} & m\tilde{\mathbf{I}}\boldsymbol{\rho}_C\boldsymbol{\omega}_{\dot{\tau}}^{\mathrm{T}} \\ m\boldsymbol{\omega}_{\dot{\tau}}(\tilde{\mathbf{I}}\boldsymbol{\rho}_C)^{\mathrm{T}} & I_A\boldsymbol{\omega}_{\dot{\tau}}\boldsymbol{\omega}_{\dot{\tau}}^{\mathrm{T}} \end{bmatrix},$$

$$\mathbf{f}^{\mathrm{inert}}(\mathbf{x},\dot{\mathbf{x}}) = \int_V \mu\boldsymbol{\Phi}^{\mathrm{T}}\dot{\boldsymbol{\Phi}}\,\mathrm{d}V\,\dot{\mathbf{x}} = -\mathbf{M}\begin{Bmatrix} \omega^2\boldsymbol{\rho}_C \\ 2\tau^{-2}(\boldsymbol{\tau}^{\mathrm{T}}\dot{\boldsymbol{\tau}})\dot{\boldsymbol{\tau}} \end{Bmatrix},$$

$$\mathbf{f}^{\mathrm{grav}}(\mathbf{x}) = \int_V \mu\boldsymbol{\Phi}^{\mathrm{T}}\mathrm{d}V\,\mathbf{g} = \begin{Bmatrix} m\mathbf{g} \\ \boldsymbol{\omega}_{\dot{\tau}}m\mathbf{g}^{\mathrm{T}}\tilde{\mathbf{I}}\boldsymbol{\rho}_C \end{Bmatrix}.$$

Here m is the body mass, $\boldsymbol{\rho}_C$ is the position of the body mass center in axes XY and I_A is the body mass inertia moment (both with respect to the attachment point A).

Attaching a rigid body to ANCF plate element

There are implementations of plate elements in ANCF proposed by various authors [4], [1]. All of them use, similarly to 2D beam elements, absolute nodal radius vectors and slopes as generalized coordinates (Fig. 3):

$$\mathbf{x} = \{\mathbf{r}^{\mathrm{T}}, \boldsymbol{\tau}_1^{\mathrm{T}}, \boldsymbol{\tau}_2^{\mathrm{T}}\}^{\mathrm{T}}. \quad (6)$$

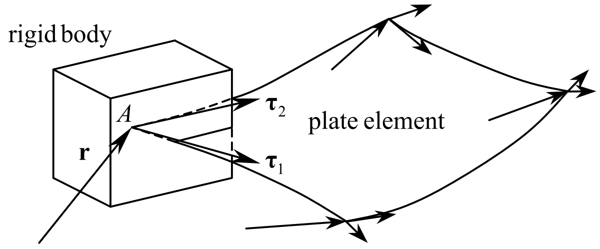

Figure 3. Rigid body attached to a plate element

In paper [8], coupling rigid body and plate was simulated using DAE technique (1). This work first handles this problem as a constraint-free one.
We use the approximate matrix of directing cosines composed of the almost orthonormal vectors: $\mathbf{A} \approx \begin{bmatrix}\boldsymbol{\tau}_1 & \boldsymbol{\tau}_2 & \boldsymbol{\tau}_3\end{bmatrix}$, where $\boldsymbol{\tau}_3 = \tilde{\boldsymbol{\tau}}_1\boldsymbol{\tau}_2$ is the normal vector. The skew-symmetric tensor of angular velocity $\tilde{\boldsymbol{\omega}} = \dot{\mathbf{A}}\mathbf{A}^{\mathrm{T}} = \dot{\boldsymbol{\tau}}_1\boldsymbol{\tau}_1^{\mathrm{T}} + \dot{\boldsymbol{\tau}}_2\boldsymbol{\tau}_2^{\mathrm{T}} + (\tilde{\boldsymbol{\tau}}_1\dot{\boldsymbol{\tau}}_2 - \tilde{\boldsymbol{\tau}}_2\dot{\boldsymbol{\tau}}_3)\boldsymbol{\tau}_3^{\mathrm{T}}$ and the matrix $\boldsymbol{\Phi} = \begin{bmatrix}\mathbf{I} & \rho_1^*\mathbf{I} - \rho_3^*\tilde{\boldsymbol{\tau}}_2 & \rho_2^*\mathbf{I} + \rho_3^*\tilde{\boldsymbol{\tau}}_1\end{bmatrix}$ are found after that. The values $\rho_k^* = \boldsymbol{\tau}_k^{\mathrm{T}}\boldsymbol{\rho}$ correspond to local components of the radius vector $\boldsymbol{\rho}$ of a point of the body.
After some cumbrous calculations we obtain equations of motion of the body in the form (5) with

$$\mathbf{M} = \begin{bmatrix} m\mathbf{I} & & \text{symm.} \\ m(\rho_1\mathbf{I} + \rho_3\tilde{\boldsymbol{\tau}}_2) & I_{11}\mathbf{I} - I_{33}\tilde{\boldsymbol{\tau}}_2\tilde{\boldsymbol{\tau}}_2 & \\ m(\rho_2\mathbf{I} - \rho_3\tilde{\boldsymbol{\tau}}_1) & I_{12}\mathbf{I} - I_{13}\tilde{\boldsymbol{\tau}}_1 - I_{23}\tilde{\boldsymbol{\tau}}_2 + I_{33}\tilde{\boldsymbol{\tau}}_1\tilde{\boldsymbol{\tau}}_2 & I_{22}\mathbf{I} - I_{33}\tilde{\boldsymbol{\tau}}_1\tilde{\boldsymbol{\tau}}_1 \end{bmatrix}$$

$$\mathbf{f}^{\mathrm{inert}} = 2\begin{bmatrix} m\rho_3\mathbf{I} \\ I_{13}\mathbf{I} + I_{33}\tilde{\boldsymbol{\tau}}_2 \\ I_{23}\mathbf{I} - I_{33}\tilde{\boldsymbol{\tau}}_1 \end{bmatrix}\dot{\tilde{\boldsymbol{\tau}}}_1\dot{\boldsymbol{\tau}}_2, \quad \mathbf{f}^{\mathrm{grav}} = \begin{bmatrix} \mathbf{I} \\ \rho_1\mathbf{I} + \rho_3\tilde{\boldsymbol{\tau}}_2 \\ \rho_2\mathbf{I} - \rho_3\tilde{\boldsymbol{\tau}}_1 \end{bmatrix} m\mathbf{g}.$$

Here ρ_k are components of the radius vector of the body mass center while I_{ij} are components of the body inertia tensor in the coordinate system with orts $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$.

CONCLUSIONS

Simulation of coupled rigid and elastic bodies can be handled as a constraint-free problem. This was shown in this paper by giving examples of beam and plate elements attached to a rigid body without relative degrees of freedom. However this technique can be generalized to any kind of joints between the bodies.

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