

A GEOMETRICAL FRAMEWORK FOR MODELING AND SIMULATION OF NONHOLONOMIC MECHANICAL SYSTEMS

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Summary. A systematic geometrical framework for effective modeling and simulation of nonholonomic systems is presented. Using this powerful means of analysis, different types of equations of motion in dependent and independent variables are obtained in compact matrix forms. Some other relevant aspects – the constraint violation problem, the involvement of independent velocities and their initial values, and the determination of constraint reactions – are also addressed. Some examples are reported.

INTRODUCTION

Many textbooks and even modern studies on analytical dynamics, and especially those devoted to the analysis of non-holonomic (NH) systems, are strongly influenced by various historical approaches and the mathematical description which are rather arduous in practical/computer applications. A large variety of formulations for NH systems [1-3] may sometimes be furthermore misleading – one can never shake off the feeling that some of them had been introduced only to solve a specific problem, and one may sometimes have difficulty in choosing the proper/best method when solving his own problem. A frequent belief is also that disparate approaches to H and NH should be used. This proves truth only in one way – the original formalism of Euler and Lagrange developed for H systems had indeed been found inapplicable to solving NH system problems [1]. The methodologies developed for NH systems are however valid for H systems as well [4-6], and a unified treatment of systems subject to H and/or NH is possible. A legitimate means of presenting these problems in a systematic way is to illustrate them geometrically [6,7].

The aim of this contribution is to present a systematic geometrical framework for effective modeling and simulation of NH systems. Different types of equations of motion in dependent and independent variables are obtained in compact matrix forms. Some other relevant aspects – the constraint violation problem, the involvement of independent velocities and their initial values, and the determination of constraint reactions – are also addressed. Some classical examples of NH systems serve as an illustration of the considerations.

MODELLING OF NONHOLONOMIC SYSTEMS

The starting point is an n -degree-of-freedom system defined in generalized coordinates $\mathbf{p} = [p_1 \dots p_n]^T$ and velocities $\mathbf{v} = [v_1 \dots v_n]^T$, whose kinematical and motion equations are $\dot{\mathbf{p}} = \mathbf{A}(\mathbf{p})\mathbf{v}$ and $\mathbf{M}(\mathbf{p})\dot{\mathbf{v}} + \mathbf{d}(\mathbf{p}, \mathbf{v}) = \mathbf{f}(\mathbf{p}, \mathbf{v}, t)$. The system can be viewed as a generalized particle on an n -dimensional manifold $\mathcal{N}$. The n -dimensional tangent space $E^n = T_p(\mathcal{N})$ to $\mathcal{N}$ at $\mathbf{p} \in \mathcal{N}$ is an Euclidean (linear vector) space, and $\mathbf{M}$ is the metric tensor matrix of the basis referred to $\mathbf{v}$ in E^n . By imposing m H and l NH constraints (bilateral and scleronomic for simplicity) on the system, $\Phi(\mathbf{q}) = \mathbf{0}$ and $\mathbf{C}_{nh}(\mathbf{p})\mathbf{v} = \mathbf{0}$, respectively, the attainable configuration is confined to a k -dimensional ($k = n - m$) manifold $\mathcal{N}_k$, while the degrees of freedom reduce to $r = n - (m + l) = k - l$. A set of independent generalized coordinates $\mathbf{q} = [q_1 \dots q_k]^T$ can then be introduced to define the system position as $\mathbf{q} \in \mathcal{N}_k$. The k -dimensional tangent space to $\mathcal{N}_k$ at $\mathbf{q} \in \mathcal{N}_k$ is then $\bar{E}^k = T_q(\mathcal{N}_k)$, and the metric tensor of the basis referred to $\dot{\mathbf{q}}$ in $\bar{E}^k$ is $\bar{\mathbf{M}} = \mathbf{D}^T \mathbf{M} \mathbf{D}$, arising in k dynamic equations $\bar{\mathbf{M}}(\mathbf{q})\ddot{\mathbf{q}} + \bar{\mathbf{d}}(\mathbf{q}, \dot{\mathbf{q}}) = \bar{\mathbf{f}}(\mathbf{q}, \dot{\mathbf{q}}, t)$ obtained using the explicit forms of H constraint equations: $\mathbf{p} = \mathbf{g}(\mathbf{q}) \Rightarrow \mathbf{v} = \mathbf{D}(\mathbf{q})\dot{\mathbf{q}} \Rightarrow \dot{\mathbf{v}} = \mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \boldsymbol{\gamma}(\mathbf{q}, \dot{\mathbf{q}})$ [6]. Due to NH constraints, $\bar{E}^k$ splits into an r -dimensional subspace $\bar{U}^r$ of admissible velocities and an l -dimensional subspace $\bar{V}^l$ where the system velocity vanishes. Introduced a set of independent velocities $\mathbf{u} = [u_1 \dots u_r]^T$ and the explicit forms of NH constraint equations: $\dot{\mathbf{q}} = \bar{\mathbf{D}}(\mathbf{q})\mathbf{u} \Rightarrow \ddot{\mathbf{q}} = \bar{\mathbf{D}}(\mathbf{q})\dot{\mathbf{u}} + \bar{\boldsymbol{\gamma}}(\mathbf{q}, \mathbf{u})$, the minimal-form governing equations of a NH system arise as $k + r$ ordinary differential equations (ODEs) $\dot{\mathbf{q}} = \bar{\mathbf{D}}(\mathbf{q})\mathbf{u}$ and $\bar{\mathbf{M}}(\mathbf{q})\dot{\mathbf{u}} + \bar{\mathbf{d}}(\mathbf{q}, \mathbf{u}) = \bar{\mathbf{f}}(\mathbf{q}, \mathbf{u}, t)$, where $\bar{\mathbf{M}} = \bar{\mathbf{D}}^T \bar{\mathbf{M}} \bar{\mathbf{D}}$ is the metric tensor matrix of the basis referred to $\mathbf{u}$ in $\bar{U}^r$.

The above concepts from differential geometry and the associated matrix formulation constitute a useful projective approach [6] to the modelling of NH systems. By applying this geometrical framework, different types of motion equations can be obtained, which, depending on the state variables used, can be grouped as follows.

- ① When using the dependent states $\mathbf{p}$ and $\mathbf{v}$, the motion equations $\dot{\mathbf{p}} = \mathbf{A}\mathbf{v}$ and $\mathbf{M}\dot{\mathbf{v}} + \mathbf{d} = \mathbf{f} - \mathbf{C}^T \boldsymbol{\lambda}$ are supplemented by the constraint equations either in the original or time-differentiated forms. The obtained differential-algebraic equations (DAEs) can either be solved directly by using the DAE solvers or, after explicit/implicit elimination of Lagrange multipliers $\boldsymbol{\lambda}$, by using the standard ODE integrators.
- ② Another possibility is to exclude H constraints/reactions from evidence and formulate the motion equations in independent coordinates $\mathbf{q}$ (the velocities $\dot{\mathbf{q}}$ are dependent). The reduced motion equations $\bar{\mathbf{M}}\ddot{\mathbf{q}} + \bar{\mathbf{d}} = \bar{\mathbf{f}} - \bar{\mathbf{C}}_{nh}^T \boldsymbol{\lambda}_{nh}$ are now supplemented by only NH constraints with equations rewritten to $\bar{\mathbf{C}}_{nh}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$. Again, the arising DAEs can be solved directly by using the DAE solvers or integrated indirectly with ODE methods.
- ③ The final possibility is to use the independent state variables $\mathbf{q}$ and $\mathbf{u}$ as defined above. The obtained minimal-form equations of motion $\dot{\mathbf{q}} = \bar{\mathbf{D}}\mathbf{u}$ and $\bar{\mathbf{M}}\dot{\mathbf{u}} + \bar{\mathbf{d}} = \bar{\mathbf{f}}$ are pure ODEs in the same number of state variables $\mathbf{q}$ and $\mathbf{u}$.

The details of these formulations and their advantages/shortcomings will be the matter of the planned presentation.

OTHER RELEVANT ASPECTS

There are at least four important aspects relevant the modeling and simulation of NH systems in practical applications.

- The constraint violation problem relates to those formulations of group ① and ② in which the constraint equations are involved in the time-differentiated forms. The problem can conveniently be solved by using the geometrical schemes developed in [8], which consist in appropriate corrections of the dependent state variables so that to eliminate the constraint violations after each step of integration or a sequence of steps.
- In formulation ③ independent velocities $\mathbf{u}$ are involved. The way they are defined, and the way the explicit forms $\dot{\mathbf{q}} = \bar{\mathbf{D}}(\mathbf{q})\mathbf{u}$ and $\ddot{\mathbf{q}} = \bar{\mathbf{D}}(\mathbf{q})\dot{\mathbf{u}} + \bar{\boldsymbol{\gamma}}(\mathbf{q}, \mathbf{u})$ of NH constraints are obtained will be explained in the presentation. The other related problem is the determination of proper initial values of $\mathbf{u}$ as these may be kinematical parameters of no physical relevance.
- In the reduction procedures leading to formulations ② and ③, respectively the H and both H and NH constraint reactions are excluded from evidence. A novel method for determination of the constraint reactions, naturally associated with the reduction procedures, will be presented. The arising codes are simple and the constraint reactions are obtained in a “resolved” forms, convenient in both symbolic manipulations and computer applications.
- A closely connected problem is “physical” formulation of constraint equations so that the associated Lagrange multipliers appearing in mathematical modelling be forces and moments in physical sense. Each constraint equation must also precisely express/describe a specified vanishing translation/velocity.

ILLUSTRATIVE EXAMPLES

Two classical examples of nonholonomic systems [1] will be reported as illustration. In the first one (Fig. 1), a sharp-edged homogeneous disc that rolls without sliding on the horizontal plane is considered. The disc is subject to one H and two NH constraints. As distinct from many standard formulations, the thickness of the disc can be included to model it as a coin, and the rolling resistance can be considered which makes the H constraint nonideal. The other example is motion of a sphere on the inside of a rough vertical cylinder (Fig. 2). Again, one H and two NH constraints are imposed on the sphere. One of the observations deduced analytically in [1] are the following features of the motion. Looking on the motion “from above” (along z axis), the sphere center moves on a circle with a constant angular speed. However, the “side-view” is so that the mass center describes a sinusoid path on the side cylinder surface. The motion corresponds to a paradoxical behavior of a basketball which, when rolling on the basket ring, often goes up and falls out of the basket. This character of the sphere motion will be approved by numerical simulation.

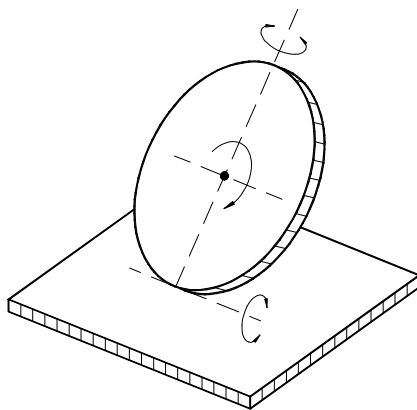


Fig. 1. The rolling disc.

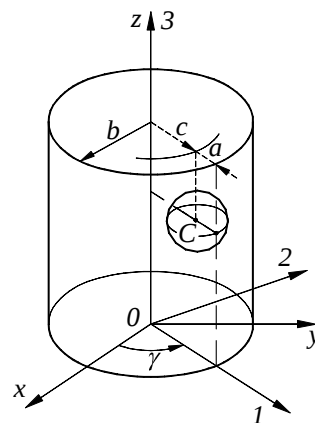


Fig. 2. A sphere moving inside a rough cylinder.

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