T-INCLUSION REGIONS FOR THE EFFECTIVE TRANSPORT COEFFICIENTS OF TWO–PHASE MEDIA

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Summary By starting from power series expanded at a number of real points and infinity we derive the fundamental inclusion relations for the special T-inclusion regions estimating in the complex domain the effective transport coefficients of two-phase media. The T- inclusion regions derived are new and the best over the entire class of rational functions and the given input data. For a particular cases they reduce to the classical complex bounds derived by Bergman [1] and Milton [2], and rederived in a different manner by Tokarzewski and Telega [4]. Nontrivial numerical examples illustrating the results obtained are also provided..

FORMULATION OF THE MATCHING PROBLEM

A macroscopic modelling of microinhomogeneous media and composites requires the evaluation of effective moduli. However their exact values are available only in specific cases; for instance in the one-dimensional periodic homogenization. In the relevant literature many papers deal with estimating of the effective coefficients of two-phase media, such as dielectric constants, magnetic permeabilities, thermal and electrical conductivities.

Assume that the macroscopic response $q_{ef}(z)/q_1$, $z = h - 1$, $h = q_2/q_1$ of two-phase composite is isotropic, where q_1 and q_2 characterize the mechanical properties of two materials the composite is made of. The following problem arises: how to determine the best bounds on $q_{ef}(z)/q_1$ in terms of its power expansions at finite number of real points $x_1, x_2, ..., x_N < \infty$ and at infinity?

It is well known that the effective transport coefficient of a two-phase medium with macroscopically isotropic symmetry has a Stieltjes integral representation given by

$$f_1(z) = \frac{Q(z)-1}{z} = \int_0^1 \frac{d\gamma_1(u)}{1+zu},\ d\gamma_1(u) > 0,\ z = h-1,\ h = \frac{q_2}{q_1},\ Q(z) = \frac{q_{ef}(z)}{q_1};\ f_1(-1) \le 1. \tag{1}$$

Let the truncated power expansions of $f_1(z)$ at $x_1,\ x_2, ...,\ x_N,\ \infty,\ -1$

$$\begin{gathered} f_1(z) = \sum_{i=0}^{p_j-1} c_{ij}(z-x_j)^i + O((z-x_j)^{p_j}),\ j = 1,2,...,N;\ \ f_1(z) = \frac{1}{z}\sum_{i=0}^{p_{N+1}-1} c_{i(N+1)}\left(\tfrac{1}{z}\right)^i + O\left(\tfrac{1}{z}\right)^{p_{N+1}} \\ f_1(z) = f_1(-1) + O(z+1);\ f_1(-1) \le 1 \end{gathered} \tag{2}$$

be given, where $-1 < x_j,\ j = 1,2,...,N+1$. We use the following notation

$$f_1(z) = f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}},\ \boldsymbol{x} = [x_1, x_2, ..., \infty, -1],\ \boldsymbol{p} = [p_1, p_2, ..., p_\infty, 1],\ f_1(-1) \le 1. \tag{3}$$

Problem *By starting from the power expansions (3), construct in a complex plane the general T-inclusion regions estimating both the Stieltjes function $f_1(z)$ and via (1_1) the effective coefficient $Q(z)$ of the two-phase composite.*

T-INCLUSION REGIONS

In the first step the inequality $f_1(-1) \le 1$ is replaced by the equality $f_1(-1) = 1$, cf. (3). The input data (3) transform to

$$f_1(z) = f_1|_{x_1,...,\infty,1}^{p_1,...,p_\infty,1}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{\boldsymbol{p}}, \quad f_1(-1) = 1. \tag{4}$$

Next by introducing $P_0 = 0$, $P_j = \sum_{i=1}^{j} p_i$, $j = 1,2,...,N$; $P = 2P_N + 1$ and using linear fractional transformation of type T we expand the given Stieltjes function $f_1(z)$ to the new Stieltjes one $f_{P.}(z)$:

$$\begin{gathered} f_{P_0+1}(z) = \frac{f_{P_0+1}(x_1)}{1+(z-x_1)f_{P_0+2}(z)},\ = \frac{f_{P_0+1}(x_1)}{1+(z-x_1)G_{P_0+2}+(z-x_1)f_{P_0+3}(z)}, ..., f_{P_1-1}(z) = \frac{f_{P_1-1}(x_1)}{1+(z-x_1)G_{P_1}+(z-x_1)f_{P_1+1}(z)}, \\ f_{P_1+1}(z) = \frac{f_{P_1+1}(x_2)}{1+(z-x_2)f_{P_1+2}(z)},\ = \frac{f_{P_1+1}(x_2)}{1+(z-x_2)G_{P_1+1}+(z-x_2)f_{P_1+2}(z)}, ..., f_{P_2-2}(z) = \frac{f_{P_2-1}(x_2)}{1+(z-x_2)G_{P_2}+(z-x_2)f_{P_2+1}(z)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{P_{N-1}+1}(z) = \frac{f_{P_{N-1}+1.}(x_{N-1})}{1+(z-x_{N-1})f_{P_{N-1}+2.}(z)},\ \cdots,\ f_{P_N-1}(z) = \frac{f_{P_N-1}(x_N)}{1+(z-x_N)f_{P-1.}(z)},\ = \frac{f_{P_N-1}(x_N)}{1+(z-x_N)G_{P-1}+(z-x_N)f_{P.}(z)}, \end{gathered} \tag{5}$$

where $G_{2i} = f_{2i}(\infty) > 0$, $i = 1,2,,...,p_\infty$ and 0 otherwise. It is convenient to rewrite (5) as follows

$$f_1(z) = f_1{}_{x_1,...,\infty,1}^{p_1,...,p_\infty,1}(z, f_P(z)),\ \text{where } f_P(z) = \int_0^1 \frac{d\gamma_P(u)}{1+zu},\ f_P(-1) = w_P. \tag{6}$$

The T-multipoint continued fraction expansion $f_1{}_{\boldsymbol{x}}^{\boldsymbol{p}}(z, f_P(z))$ of $f_1(z)$ matches the input data (4) for any Stieltjes functions $f_P(z) = \int_0^1 \frac{d\gamma_P(u)}{1+zu}$ satisfying $\int_0^1 \frac{d\gamma_P(u)}{1-u} = w_P$. On account of that the T-inclusion region

$\Phi_P(z)$ should incorporate all admissible values of the Stieltjes functions $f_1(z)$ matching the input data (3). Hence we have

$$
\begin{gathered}
f_1(z) \in \Phi_P(z) = \{F_P(z,u);\ 0 \le v \le 1,\ -1 \le u \le 1\}, \\
F_P(z,u) = f_1{}^{p_1,\ldots,p_\infty,1}_{x_1,\ldots,\infty,1}(z, vw_p F_0(z,u)),\ F_0(z,u) = (1+u \text{ if } -1 \le u \le \text{ and } \tfrac{1-u}{1+zu} \text{ if } 0 \le u \le 1).
\end{gathered} \tag{7}
$$

Here $F_0(z,u)$ is called the elementary bounding function. It consists of two lines, the straight line $1+u$ if $-1 \le u \le 0$ and the arc of the circle $(1-u)/(1+zu)$ if $0 \le u \le 1$, see (7_2).

FUNDAMENTAL RELATIONS FOR T-INCLUSION REGIONS

Now we are prepared to formulate the main result of this contribution reading by the following theorem:.
Theorem 1. *Let the truncated power expansions of Stieltjes function* $f_1(z)$

$$
f_1(z) = f_1|_{\boldsymbol{x}}^{p(k)}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{p(k)},\quad \boldsymbol{x} = [x_1, x_2, ..., x_N, \infty, -1],\ \boldsymbol{p}(k) = [p_1(k), p_2(k), ..., p_\infty(k), 1] \tag{8}
$$

be given, where $k = \sum_{j=1}^{N} p_j + p_\infty + 1$*. Then T-inclusion regions* $\Phi_{p(k)}(z)$ *generated by the truncated series (8) satisfy the following fundamental relations*

$$
f_1(z) \in \Phi_{P(k)}(z) \subset \Phi_{P(k-1)}(z),\ k = 2, 3, \ldots,. \tag{9}
$$

provided $p(k) \le p(k+1),\ k = 1, 2, ...,$ *i.e.* $p_j(k+1) - p_j(k) \ge 0,\ j = 1, 2, ...$ *These inclusion relations imply that the best* $\Phi_{P(k)}(z)$ *estimate of* $f_1(z)$ *is obtained using only the given number of power series coefficients (k is fixed) and that the use of additional coefficients (higher k) improves the estimate* $\Phi_{P(k)}(z)$.

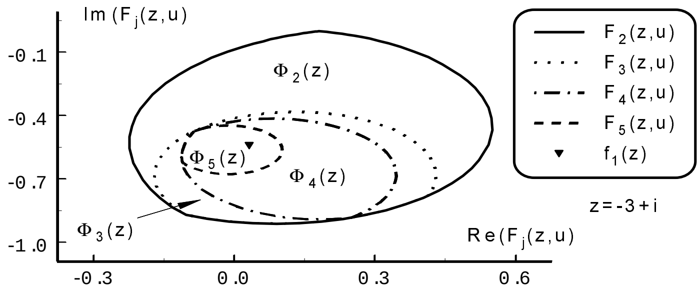

Fig. 1: Monotonic sequence of T-inclusion regions generated by the power series $f_1|_{\boldsymbol{x}}^{\boldsymbol{p}}(z-\boldsymbol{x}) + O(z-\boldsymbol{x})^{p(k)}$, $\boldsymbol{x}= [0, 3, \infty, -1]$ $\boldsymbol{p}(2) = [1, 0, 0, 1]$, $\boldsymbol{p}(3) = [1, 0, 1, 1]$, $\boldsymbol{p}(4) = [1, 1, 1, 1]$, $\boldsymbol{p}(5) = [1, 1, 2, 1]$ representing the Stieltjes function $f_1(z) = \dfrac{1}{z}\left(1 + \frac{2.5}{z} \ln \frac{1.2+0.1z}{2+0.5z}\right)$, cf. (9)

For $p_\infty = 0$ we have $G_{2i} = 0,\ i = 1, 2, ...,\ p_\infty$, see (3) and (5). For such a case the T-continued fraction expansion of $f_1(z)$ transforms to the S-continued fraction one reported in mathematical literature. Moreover, the substitution $p_1 = 1, p_2 = 1, ..., p_\infty = 0$ reduces the matching problem (3) to the fitting one investigated by Bergman [1] and Milton [2], see also Tokarzewski *et al* [3.4].

FINAL REMARKS

By starting from the several truncated power series we derive, using special T-multipoint continued fraction technique, the general T-inclusion regions estimating in a complex domain the effective transport coefficients $Q(z)$ of two-phase media such as dielectric or diffusion constants, thermal or electrical conductivities, magnetic permeabilities. The T-inclusion regions obtained are new and furnish the best estimates of $Q(z)$. In special cases they reduce to the known complex bounds. Numerical examples exhibiting the usefulness of the results obtained are also provided.

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