# ANALYSIS AND DESIGN OF DISPERSIVE MATERIALS AND STRUCTURES

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*Summary* First, a multiscale assumed strain variational formulation is developed for the prediction of frequency spectra of periodic materials. This approach facilitates the generation of reduced order models using mode projections, hence practically providing up to an order of magnitude decrease in computational problem size compared to direct analysis. Second, computed frequency band structures are tailored to yield synthesized composite materials with desired dynamic characteristics. The designed materials are then used to form structures, at a larger length scale, for a variety of purposes. The generated composite structures are significantly more amenable to manufacturing compared to those generated via traditional topology optimization. Applications include high-frequency vibration isolators and structural waveguides.

## FREQUENCY RESPONSE OF PERIODIC MATERIALS

Within periodically heterogeneous media, wave scattering and dispersion take place across constituent material interfaces in such a way that an overall wave attenuation effect arises at certain frequency ranges known as *stop bands*. This phenomenon, which is attributed to a mechanism of destructive interference among the scattered wave field, can be utilized in designing composite materials and structures with tailored frequency-dependent dynamic characteristics. A necessary tool for this design process is an accurate and efficient analysis technique for computing band diagrams (dispersion curves) for periodic materials.

### Multiscale assumed strain variational formulation

A local (unit cell) domain Y is considered as part of a global periodic bounded structure Ω, where the ratio $\varepsilon$ between the length scales of the domains is assumed to be large (i.e., $\varepsilon$ is a small number). Based on this multiscale construct, the following two-field variational problem is defined:

Problem 1: *Find* $\mathbf{u} \in S_{\Omega\times Y}$ and $\boldsymbol{\varepsilon} \in E_{\Omega\times Y}$ *such that*

$$\int_\Omega \langle (\nabla_{\mathbf{x}} \mathbf{w} + \frac{1}{\varepsilon}\nabla_{\mathbf{y}} \mathbf{w}) : \mathbf{C} : \boldsymbol{\varepsilon}\rangle d\Omega + \int_\Omega \langle \rho \mathbf{w}\cdot \frac{\partial^2 \mathbf{u}}{\partial t^2}\rangle d\Omega - \langle G_{\mathrm{ext}}(\mathbf{w})\rangle + \int_\Omega \langle \boldsymbol{\gamma} : \mathbf{C} : (\nabla_{\mathbf{x}} \mathbf{u} + \frac{1}{\varepsilon}\nabla_{\mathbf{y}} \mathbf{u} - \boldsymbol{\varepsilon})\rangle d\Omega = 0, \qquad (1)$$

where $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ denote displacement and strain, respectively, and $\mathbf{w}$ and $\boldsymbol{\gamma}$ are the weighting functions associated to each, and $\rho$, $\mathbf{C}$ and $G_{\mathrm{ext}}$ denote density, elasticity tensor and virtual work done by external forces, respectively. Based on McDevitt *et al.*[1], the displacement and strain fields are assumed to take the following forms:

$$\mathbf{u}(\mathbf{x},\mathbf{y},t) = \bar{\mathbf{u}}(\mathbf{x},t) + \varepsilon \mathbf{P}(\mathbf{y})\cdot \boldsymbol{\phi}(\mathbf{x},t), \qquad (2)$$

$$\boldsymbol{\varepsilon}(\mathbf{x},\mathbf{y},t) = (\nabla_{\mathbf{x}}\bar{\mathbf{u}} + \nabla_{\mathbf{y}}(\mathbf{P}\cdot\boldsymbol{\psi}))^S, \qquad (3)$$

where $\bar{\mathbf{u}}$ represents an averaged macroscale displacement, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are macroscale vectors of generalized coordinates, and $\mathbf{P}$ is a 'projection matrix'. Substituting (2)-(3) into (1) yields a set of Euler-Lagrange equations. Subsequently, assuming a Floquet-Bloch solution results in the following dispersion relation eigenvalue problem:

$$\mathbf{K}^*(\mathbf{k})\{\overline{\mathbf{U}}^{\mathrm{T}} \quad \boldsymbol{\Phi}^{\mathrm{T}} \quad \boldsymbol{\Psi}^{\mathrm{T}}\}^{\mathrm{T}} - \omega^2\mathbf{M}^*\{\overline{\mathbf{U}}^{\mathrm{T}} \quad \boldsymbol{\Phi}^{\mathrm{T}} \quad \boldsymbol{\Psi}^{\mathrm{T}}\}^{\mathrm{T}} = \mathbf{0} \qquad (4)$$

The central development in the present work is the choice of functions (vectors) for projection matrix $\mathbf{P}$. A selected set of Floquet-Bloch eigenvectors based on an assessment of the modal contribution of the eigenvectors in both the temporal and spatial frequency domains is used to form $\mathbf{P}$.

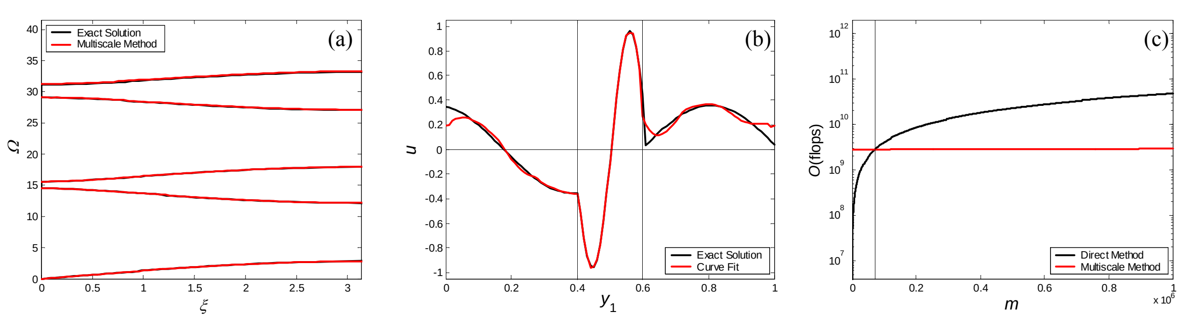



**Figure 1.** (a) Frequency spectrum (dispersion curves) for *P*-mode wave propagation normal to the layering. Non-dimensional frequency $\Omega$ and wavenumber $\xi$ define the ordinate and abscissa, respectively. Insert: schematic of unit cell, where black color represents stiff material and white color represents compliant material. (b) Least square fit of a particular exact mode shape onto projection subspace $\mathbf{P}$. (c) Computational effort (order of operation count) for multiscale and direct methods ($m$ denotes the size of the matrix problem, which in turn is dependent on the resolution of the finite element mesh). The vertical line identifies the break-even point among the two methods.

**Accuracy and efficiency considerations**

To test the accuracy and efficiency of the developed multiscale method, several analyses were conducted on a simple 1D bi-material layered material whose exact solution is readily available. Fig. 1a shows an excellent agreement of computed dispersion curves with the exact solution. Fig. 1b demonstrates the richness of the chosen projection space through curve fitting a particular mode shape. Finally, in Fig 1c, the efficiency of the multiscale reduced order method is compared to that of a direct method, indicating significant reduction in *flops* for large systems.

## DESIGN METHODOLOGY

The technique mentioned above is employed as an analysis tool for designing periodic materials and structures with desired dynamic characteristics. The design methodology (see ref. (2) for 1D problems) also has a multiscale character.

**Design of fine-scale unit cell**

The unit cell constitutes a "building block" from which the global structure is to be formed. It is designed in such a way as to have stop bands located in predetermined locations in the frequency domain. These locations are selected in accordance to the desired function of the cell within the bounded structure. Figure 2 illustrates a designed 2D unit cell and shows a stability diagram that is used for material selection. The resulting frequency spectra (for *SH* and *P-SV* wave motion) of the designed periodic material are also shown.

**Design of coarse-scale structure**

The designed unit is used to form a bounded composite structure. As an example, a waveguiding structure (which can employed as a vibration isolator) is designed using the proposed methodology. The aim is to have the structure steer mechanical waves along a prescribed path at a particular frequency. The unit cell from which the guide "wall" is built is designed to have this frequency lie within its first stop band. Figure 3 shows the wave field conforming to the guide as desired. Compared to structures designed by traditional topology optimization, this structure has the advantage that its material layout geometric features are simple and hence amenable to manufacturing.

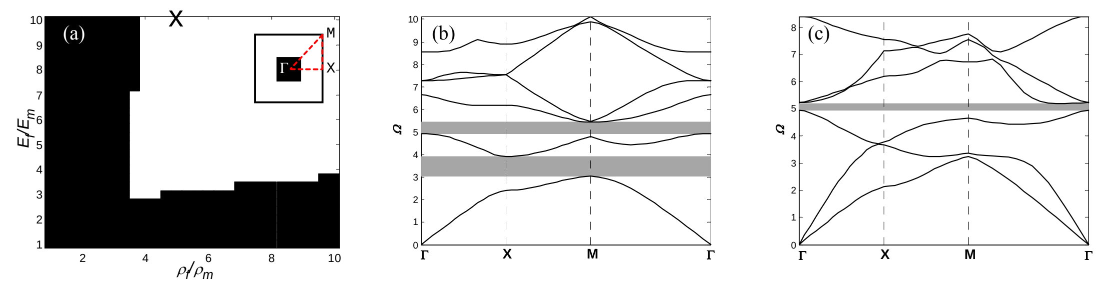
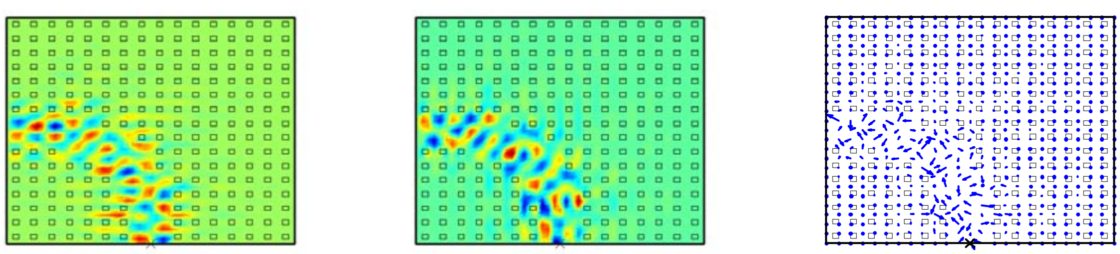



**Figure 2.** (a) Contrasting material properties stability map for the designed unit cell (for *SH* wave motion). $E$ and $\rho$ denote Young's modulus and density, respectively. $(\cdot)_f$ and $(\cdot)_m$ refer to stiff (fiber) and compliant (matrix) materials, respectively. The black zone corresponds to a pass band and the white zone corresponds to a stop band. "x" mark indicates choice of ratios of material properties. Insert: Designed unit cell material layout. Simple square configuration is chosen for ease of manufacture. The irreducible Brillouin zone is shown in dashed lines. Points marked Γ, X and M correspond to the following wave vectors: $\mathbf{k} = [0, 0]$ at Γ, $\mathbf{k} = [\pi/d, 0]$ at X, and $\mathbf{k} = [\pi/d, \pi/d]$ at M, where $d$ is the unit cell side length. Frequency spectrum for chosen design is shown for (b) out-of-plane *SH*, and (c) in-plane coupled *P-SV*, wave propagation. In (b)-(c), the abscissa is defined by the magnitude of the wave vector whose end point sweeps the paths Γ>X, X>M, and M>Γ, respectively. Stop bands are shaded in grey.



(a) (b) (c)

**Figure 3.** Dynamic response due to excitation at point "x". The forcing load is in-plane and perpendicular to the edge of the structure. (a) x-component, and (b) y-component, of displacement wave field. (c) Resultant displacement vector field indicating "vorticial-type" wave motion. Excitation frequency lies at the center of the first stop band corresponding to the designed unit cell for in-plane wave motion (see Fig. 2c). Note, that this unit cell is used to form the "wall" of the waveguide.

**References**

[1] McDevitt, T.W., Hulbert, G.M., and Kikuchi, N.: An Assumed Strain Method for the Dispersive Global-Local Modeling of Periodic Structures. *Computer Methods in Applied Mechanics and Engineering*, **190**:6425-6440, 2001.

[2] Hussein, M.I., Hulbert, G.M., and Scott, R.A.: Tailoring of Wave Propagation Characteristics in Periodic Structures with Multilayer Unit Cells," *17th American Society of Composites Technical Conference* [CD-ROM], West Lafayette, Indiana, October 2002.