DISPERSION AND STABILITY ANALYSIS OF WAVES IN PRE-STRESSED IMPERFECTLY BONDED LAYERED COMPOSITES

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Summary The dispersive behaviour and stability issues of time-harmonic waves propagating in an imperfectly bonded pre-stressed symmetric layered composite, which consists of incompressible isotropic elastic materials is considered. The shear spring type resistance model is employed to simulate the imperfect interface. Dispersion relations for both flexural and extensional waves are obtained. The stability criteria are discussed and the neutral curves are plotted.

BACKGROUND AND FORMULATION OF THE PROBLEM

Wave propagation in pre-stressed layered composites has been studied in [1, 2] for the perfectly bonded case. In the present paper, recent work done by the authors on the more general case of imperfectly bonded composite [3, 4] is discussed.

The bimaterial pre-stressed symmetric layered composite consists of an inner layer (thickness $2d$) and two identical outer layers (thickness h) of isotropic incompressible elastic materials, where the principle axes of strain in each layer are coincident. The Cartesian coordinate system is chosen such that the origin O is at the mid-plane of the composite, x_1 and x_2-axes are also coincident with the principal axes, the x_2-direction is normal to the free surface of the layered composite and time harmonic wave propagation is in x_1-direction. The outer layers and inner layer are homogeneous with material parameters and mass density $\alpha,\ \beta,\ \gamma,\ \rho$ and $\alpha^*,\ \beta^*,\ \gamma^*,\ \rho^*$, respectively. In the remainder of this paper, all quantities with an asterisk refer to variables and parameters of the inner layer.

Harmonic wave propagating in the outer layer are expressed as $(u_1,u_2,p) = (A_1,A_2,kP)e^{qkx_2}e^{ik(x_1-vt)}$, where u_i is the displacement increment in x_i-direction, p is pressure increment, k is wavenumber, v is phase speed, t is time, A_i, P are arbitrary constants and q is calculated from $q^4 + (\xi - 2\overline{\beta})q^2 + (\overline{\alpha} - \xi) = 0$, where $\overline{\beta} = \beta/\gamma$, $\overline{\alpha} = \alpha/\gamma$ and $\xi = \rho v^2/\gamma$ [3]. The corresponding solution for the inner layer will yield $q^{*4} + (\xi^* - 2\overline{\beta}^*)q^{*2} + (\overline{\alpha}^* - \xi^*) = 0$, where $\xi^* = a\xi$ and $a = r\rho^*/\rho,\ r = \gamma/\gamma^*$.

Due to the symmetric geometry of the composite, only the upper half of the composite needs to be considered. The incremental traction free surface conditions are $s_{021}(x_1,d+h,t) = s_{022}(x_1,d+h,t) = 0$ where s_{02i} are the nominal stress increment components. The mid-plane conditions for flexural waves are $u_1^*(x_1,0,t) = s_{022}^*(x_1,0,t) = 0$, and for extensional waves are $u_2^*(x_1,0,t) = s_{021}^*(x_1,0,t) = 0$. At the interface, the continuity conditions are $s_{021}^*(x_1,d,t) = s_{021}(x_1,d,t)$, $s_{022}^*(x_1,d,t) = s_{022}(x_1,d,t)$, $u_2^*(x_1,d,t) = u_2(x_1,d,t)$ and the incremental nominal shear stress is $s_{021}^*(x_1,d,t) = k_x(\gamma/h)$ $[u_1(x_1,d,t) - u_1^*(x_1,d,t)]$, where k_x is the non-dimensional shear spring parameter.

DISPERSION RELATIONS AND STABILITY CONSIDERATIONS

For flexural waves the dispersion relation is obtained as

$$\begin{aligned} &q_1^* S_1^* C_2^* \left[f^*(q_2^*)^2 \hat{\Delta}_1 + f^*(q_2^*)\hat{\Delta}_2 + \hat{\Delta}_3 \right] - q_2^* C_1^* S_2^* \left[f^*(q_1^*)^2 \hat{\Delta}_1 + f^*(q_1^*)\hat{\Delta}_2 + \hat{\Delta}_3 \right] - \left[f^*(q_1^*) - f^*(q_2^*) \right]\left[q_1^* q_2^* S_1^* S_2^* \hat{\Delta}_4 + C_1^* C_2^* \hat{\Delta}_5 \right] \\ &+ \left(\frac{kh}{rk_x}\right)\left\{ \left[f^*(q_1^*) - f^*(q_2^*) \right] C_1^* C_2^* \hat{\Delta}_3 + \left[q_1^* f^*(q_2^*)^2 S_1^* C_2^* - q_2^* f^*(q_1^*)^2 C_1^* S_2^* \right]\hat{\Delta}_4 \right\} = 0, \end{aligned} \tag{1}$$

where

$$\hat{\Delta}_1 = 2q_1q_2 f(q_1)f(q_2) - q_1q_2\left[f(q_1)^2 + f(q_2)^2\right]C_1C_2 + \left[q_1^2 f(q_2)^2 + q_2^2 f(q_1)^2\right]S_1S_2,$$

$$\hat{\Delta}_2 = 2r\left\{q_1q_2 f(q_1)f(q_2)[f(q_1)+f(q_2)](C_1C_2-1) - \left[q_1^2 f(q_2)^3 + q_2^2 f(q_1)^3\right]S_1S_2\right\},$$

$$\hat{\Delta}_3 = r^2\left\{-2q_1q_2 f(q_1)^2 f(q_2)^2 (C_1C_2-1) + \left[q_1^2 f(q_2)^4 + q_2^2 f(q_1)^4\right]S_1S_2\right\},$$

$$\hat{\Delta}_4 = r[f(q_1)-f(q_2)]\left[q_1 f(q_2)^2 C_1S_2 - q_2 f(q_1)^2 S_1C_2\right], \quad \hat{\Delta}_5 = rq_1q_2\left[f(q_1)-f(q_2)\right]\left[q_1 f(q_2)^2 S_1C_2 - q_2 f(q_1)^2 C_1S_2\right],$$

$$C_m = \cosh(q_m kh), \quad S_m = \sinh(q_m kh), \quad C_m^* = \cosh(q_m^* Dkh), \quad S_m^* = \sinh(q_m^* Dkh) \quad (m = 1,2),$$

$$f(q_m) = 1 + q_m^2 - \sigma, \quad f^*(q_m^*) = 1 + q_m^{*2} - r\sigma, \quad \sigma = \sigma_2/\gamma, \quad D = d/h.$$

Equation (1) reduces to the dispersion relation of the perfectly bonded case [1] when $k_x \to \infty$, while it reduces to the fully slipping interface case when $k_x = 0$. The dispersion relation for extensional waves is obtained from Eq. (1) by replacing C_m^* by S_m^* and S_m^* by C_m^* ($m = 1,2$).

For a particular mode when $\xi < 0$, the phase speed v is pure imaginary and instead of harmonic wave propagation in x_1-direction, standing waves with amplitudes that grow exponentially with time are produced i.e., an *unstable* state. On the other hand, harmonic waves travelling in x_1-direction will be *stable* for a particular mode when $\xi > 0$ while $\xi = 0$ corresponds to the *neutral* state. By setting $\xi = 0$, Eq. (1) yields the bifurcation equation, which is a quartic equation of σ i.e.,

$$\Phi_4^{(\mathrm{F})}\sigma^4 + \Phi_3^{(\mathrm{F})}\sigma^3 + \Phi_2^{(\mathrm{F})}\sigma^2 + \Phi_1^{(\mathrm{F})}\sigma + \Phi_0^{(\mathrm{F})} = 0, \tag{2}$$

where the coefficients $\Phi_i^{(\mathrm{F})}$ $(i = 1,...,4)$ are functions of kh, k_x, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}^*$, $\bar{\beta}^*$, r, a and D. The neutral curves, used to separate the stable and unstable region can be obtained from Eq. (2). This equation will reduce to a quadratic equation of σ when $k_x \to \infty$, while it remains as a quartic equation when $k_x = 0$. Hence, in general four branches of neutral curves are expected for an imperfect interface and fully slipping cases while a maximum of only two branches could occur for perfectly bonded case. Similar equations for extensional waves are discussed in [4].

NUMERICAL RESULTS

One of the Examples considered in [3, 4] will be briefly discussed here. The outer and inner layers are Mooney-Rivlin and Varga materials and the primary deformation of both layers are plane strain with principle stretches $\lambda_1 = \lambda_2^{-1} = 1.25$ and $\lambda_1^* = \lambda_2^{*-1} = 2.25$ in which yields $\bar{\alpha} = 2.441$, $\bar{\beta} = 1.721$, $\bar{\alpha}^* = 25.629$, and $\bar{\beta}^* = 5.063$. The other prescribed parameters are $r = 2.5$, $a = 20$, $\sigma = 0$ and $D = 1$. The dispersion curves and neutral curves are shown in Figs. 2 and 3 where the limiting values as $kh \to 0$ and $kh \to \infty$ are given in [4].

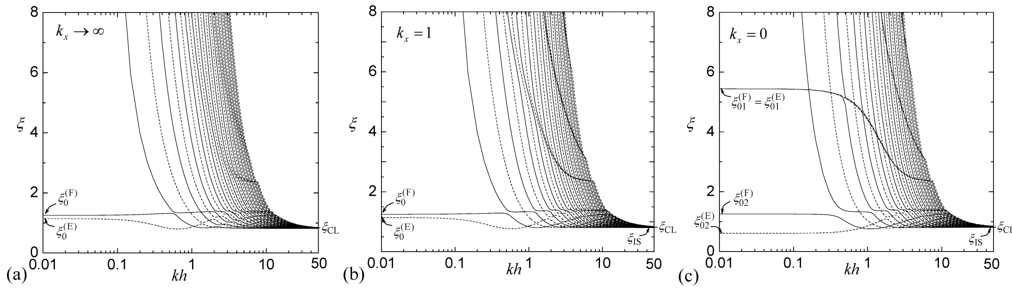

Fig. 2. Non-dimensional squared phase speed ξ ; solid lines - flexural waves and dashed lines - extensional waves.

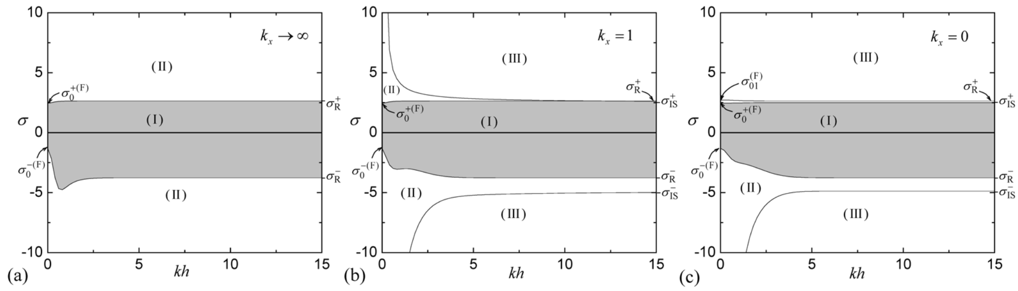

Fig. 3. Neutral curves for flexural waves; shaded area is the region (I) where all modes are stable, region (II) is where the fundamental mode is unstable and region (III) is where the fundamental and the next lowest modes are unstable.

CONCLUSIONS

The dispersive behaviour, for flexural and extensional waves are similar at low and high wavenumber limits. For each type of wave at low wavenumber limit, for $k_x > 0$ only one finite limiting squared phase speed ($\xi_0^{(\mathrm{F})}$ or $\xi_0^{(\mathrm{E})}$) exists while for $k_x = 0$ two finite limiting squared phase speeds ($\xi_{01}^{(\mathrm{F})}, \xi_{02}^{(\mathrm{F})}$ or $\xi_{01}^{(\mathrm{E})}, \xi_{02}^{(\mathrm{E})}$) are found. At high wavenumber limit, both flexural and extensional waves tend to the same limits, which may be squared phase speeds of surface waves (ξ_R), interfacial waves (ξ_{IS} or ξ_{IP}), and limiting phase speed of the composite (ξ_{CL}). The same stable ranges of σ are found for perfectly bonded and imperfect interface cases at the low wave number limit, while at the high wavenumber limit the stable range is the same for imperfect interface and fully slipping interface cases.

References

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