THREE-DIMENSIONAL TRANSIENT THERMOELASTIC ANALYSIS OF ORTHOTROPIC FUNCTIONALLY GRADED RECTANGULAR PLATE

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Summary This paper is concerned with the transient thermoelastic analysis of an orthotropic functionally graded rectangular plate due to nonuniform heat supply. The material constants of the orthotropic rectangular plate are assumed to vary exponentially in the thickness direction. We obtain the exact solution for the three-dimensional temperature change in a transient state, and three-dimensional thermal stresses of a simple supported rectangular plate.

INTRODUCTION

Functionally graded materials (FGM) have been developed as a new material that is adaptable for a super-high-temperature environment. It is well-known that thermal stress distributions in a transient state can show large values compared with the one in a steady state. Therefore, the transient thermoelastic problems become important. On the other hand, the exact treatments for thermoelastic problems of FGM have been done thus circumventing the laminate theory approximation. The reports concerned with the three-dimensional transient thermoelastic problems of FGM are few. Recently, Vel and Batra [1] treated a simply supported functionally graded rectangular plate, which material properties are represented by a Taylor series expansion.

In the present paper, we analyzed exactly the three-dimensional transient problem of thermoelasticity involving an orthotropic functionally graded rectangular plate due to nonuniform heat supply. The material properties of the orthotropic rectangular plate are assumed to vary exponentially in the thickness direction.

ANALYSIS

We consider an orthotropic functionally graded rectangular plate that has nonhomogeneous thermal and mechanical properties in the thickness direction. The thickness and the lengths of the sides of it are represented by B, $2L_x$ and $2L_y$, respectively. We assume that the rectangular plate is initially at zero temperature and is suddenly heated from the lower and upper surfaces by surrounding media with relative heat transfer coefficients h_a and h_b. We denote the temperatures of the surrounding media by the functions $T_a f_a(x) g_a(y)$ and $T_b f_b(x) g_b(y)$ and assume its end surfaces are held zero temperature. The transient heat conduction equation is taken in the following form:

$$\frac{\partial}{\partial x}\left\{\lambda_x(z)\frac{\partial T}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\lambda_y(z)\frac{\partial T}{\partial y}\right\}+\frac{\partial}{\partial z}\left\{\lambda_z(z)\frac{\partial T}{\partial z}\right\}=c\rho\frac{\partial T}{\partial t} \tag{1}$$

The thermal conductivities is assumed to take the following forms:

$$\lambda_i(z)=\lambda_{i0}\, exp(az/B)\,,\quad i=x,y,z \tag{2}$$

while the specific heat c and density ρ are constant. In Eq.(2), a is an arbitrary constant which is not zero. Substituting the Eq.(2) into the Eq.(1), the transient heat conduction equation in dimensionless form is

$$\bar{\lambda}_{x0}\frac{\partial^2\bar{T}}{\partial\bar{x}^2}+\bar{\lambda}_{y0}\frac{\partial^2\bar{T}}{\partial\bar{y}^2}+a\frac{\partial\bar{T}}{\partial\bar{z}}+\frac{\partial^2\bar{T}}{\partial\bar{z}^2}=e^{-a\bar{z}}\frac{\partial\bar{T}}{\partial\tau} \tag{3}$$

For the sake of brevity, we introduce that the temperature functions $f_a(x)$, $f_b(x)$, $g_a(y)$ and $g_b(y)$ are distributed symmetrically in x and y axes. Introducing the finite cosine transformations with respect to the variable $\bar{x}$ and $\bar{y}$ and Laplace transformation with respect to the variable τ, the solution of Eq.(3) can be obtained as follows:

$$\bar{T}=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\bar{T}_{km}(\bar{z},\tau)\cos q_k\bar{x}\cos s_m\bar{y} \tag{4}$$

where

$$\bar{T}_{km}(\bar{z},\tau)=\frac{4}{\bar{L}_x\bar{L}_y}\left[\frac{1}{F}\left\{\bar{A}'exp\left[-\frac{a}{2}(1-\gamma)\bar{z}\right]+\bar{B}'exp\left[-\frac{a}{2}(1+\gamma)\bar{z}\right]\right\}\right.$$
$$\left.-\sum_{j=1}^{\infty}\frac{2\,exp(-\frac{a}{2}\bar{z}-\frac{a^2\mu_j^2}{4}\tau)}{\mu_j\Delta'(\mu_j)}\left\{\bar{A}J_\gamma\left[\mu_j\, exp(-\frac{a}{2}\bar{z})\right]+\bar{B}Y_\gamma\left[\mu_j\, exp(-\frac{a}{2}\bar{z})\right]\right\}\right] \tag{5}$$

In Eqs.(4) and (5), q_k, s_m and γ are

$$q_k = \frac{(2k-1)\pi}{2\bar{L}_x}, \quad s_m = \frac{(2m-1)\pi}{2\bar{L}_y}, \quad \gamma = \sqrt{1+\frac{4}{a^2}(q_k^2\bar{\lambda}_{x0} + s_m^2\bar{\lambda}_{y0})} \tag{6}$$

We now analyze the transient thermoelasticity of an orthotropic functionally graded rectangular plate with simply supported edges as a three-dimensional problem. The elastic stiffness constant C_{ij} and the coefficient of linear thermal expansion α_i are assumed to take the following forms:

$$\bar{C}_{ij}(\bar{z}) = \bar{C}_{ij}^0 \, exp(l\bar{z}), \quad \bar{\alpha}_i(\bar{z}) = \bar{\alpha}_{i0} \, exp(b\bar{z}) \tag{7}$$

where l and b are arbitrary constants. Substituting the displacement-strain relations, the stress-strain relations and Eq.(7) into the equilibrium equations, the displacement equations of equilibrium are written as

$$\bar{C}_{11}^0\bar{u}_{,\bar{x}\bar{x}} + \bar{C}_{66}^0\bar{u}_{,\bar{y}\bar{y}} + \bar{C}_{55}^0\bar{u}_{,\bar{z}\bar{z}} + l\bar{C}_{55}^0\bar{u}_{,\bar{z}} + (\bar{C}_{12}^0 + \bar{C}_{66}^0)\bar{v}_{,\bar{x}\bar{y}} + (\bar{C}_{13}^0 + \bar{C}_{55}^0)\bar{w}_{,\bar{x}\bar{z}} + l\bar{C}_{55}^0\bar{w}_{,\bar{x}} = \bar{\beta}_x^0 \, exp(b\bar{z})\bar{T}_{,\bar{x}} \tag{8}$$

$$(\bar{C}_{12}^0 + \bar{C}_{66}^0)\bar{u}_{,\bar{x}\bar{y}} + \bar{C}_{66}^0\bar{v}_{,\bar{x}\bar{x}} + \bar{C}_{22}^0\bar{v}_{,\bar{y}\bar{y}} + \bar{C}_{44}^0\bar{v}_{,\bar{z}\bar{z}} + l\bar{C}_{44}^0\bar{v}_{,\bar{z}} + (\bar{C}_{23}^0 + \bar{C}_{44}^0)\bar{w}_{,\bar{y}\bar{z}} + l\bar{C}_{44}^0\bar{w}_{,\bar{y}} = \bar{\beta}_y^0 \, exp(b\bar{z})\bar{T}_{,\bar{y}} \tag{9}$$

$$(\bar{C}_{13}^0 + \bar{C}_{55}^0)\bar{u}_{,\bar{x}\bar{z}} + l\bar{C}_{13}^0\bar{u}_{,\bar{x}} + (\bar{C}_{23}^0 + \bar{C}_{44}^0)\bar{v}_{,\bar{y}\bar{z}} + l\bar{C}_{23}^0\bar{v}_{,\bar{y}} + \bar{C}_{55}^0\bar{w}_{,\bar{x}\bar{x}} + \bar{C}_{44}^0\bar{w}_{,\bar{y}\bar{y}} + \bar{C}_{33}^0\bar{w}_{,\bar{z}\bar{z}} + l\bar{C}_{33}^0\bar{w}_{,\bar{z}}$$
$$= \bar{\beta}_z^0 \, exp(b\bar{z})[(l+b)\bar{T} + \bar{T}_{,\bar{z}}] \tag{10}$$

If the lower and upper surfaces are traction free, the boundary conditions of lower and upper surfaces can be represented as follows:

$$\bar{z} = 0, \, 1; \quad \bar{\sigma}_{zz} = 0, \quad \bar{\sigma}_{yz} = 0, \quad \bar{\sigma}_{zx} = 0 \tag{11}$$

We now consider the case of a simply supported rectangular plate.

$$\bar{x} = \pm\bar{L}_x; \quad \bar{\sigma}_{xx} = 0, \quad \bar{v} = 0, \quad \bar{w} = 0 \tag{12}$$

$$\bar{y} = \pm\bar{L}_y; \quad \bar{\sigma}_{yy} = 0, \quad \bar{u} = 0, \quad \bar{w} = 0 \tag{13}$$

We assume the solutions of Eqs.(8)- (10) in order to satisfy Eqs.(12) and (13) in the following form.

$$\bar{u} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left[U_{ckm}(\bar{z}) + U_{pkm}(\bar{z})\right] sin\, q_k\bar{x}\, cos\, s_m\bar{y}, \quad \bar{v} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left[V_{ckm}(\bar{z}) + V_{pkm}(\bar{z})\right] cos\, q_k\bar{x}\, sin\, s_m\bar{y},$$

$$\bar{w} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left[W_{ckm}(\bar{z}) + W_{pkm}(\bar{z})\right] cos\, q_k\bar{x}\, cos\, s_m\bar{y} \tag{14}$$

In expressions (14), the first term on the right side gives the homogeneous solution and the second term of right side gives the particular solution. In order to obtain the homogeneous solution, we assume that

$$\left[U_{ckm}(\bar{z}), V_{ckm}(\bar{z}), W_{ckm}(\bar{z})\right] = (U_{ckm}^0, V_{ckm}^0, W_{ckm}^0)\, exp(\lambda\bar{z}) \tag{15}$$

The details of the homogeneous solution are omitted here for the sake of brevity. In order to obtain the particular solution, we use the series expansions of the Bessel functions. Since the order γ of the Bessel function in Eq.(5) is not integer in general, Eq.(5) can be written as the following expression.

$$\bar{T}_{km}(\bar{z},\tau) = \sum_{n=0}^{\infty}\left\{a_{n1}(\tau)\, exp\left[-\frac{a}{2}(2n+1+\gamma)\bar{z}\right] + a_{n2}(\tau)\, exp\left[-\frac{a}{2}(2n+1-\gamma)\bar{z}\right]\right\} \tag{16}$$

$U_{pkm}(\bar{z})$, $V_{pkm}(\bar{z})$ and $W_{pkm}(\bar{z})$ of the particular solution are obtained as the function system like Eq.(16).

In expressions (3)-(16), we have introduced the following dimensionless values:

$$\bar{T} = T/T_0, \quad (\bar{x},\bar{y},\bar{z},\bar{L}_x,\bar{L}_y) = (x,y,z,L_x,L_y)/B, \quad (\bar{\lambda}_{x0},\bar{\lambda}_{y0}) = (\lambda_{x0},\lambda_{y0})/\lambda_{z0}, \tau = \kappa_{z0}t/B^2,$$
$$\kappa_{z0} = \lambda_{z0}/(c\rho), \quad \bar{C}_{ij} = C_{ij}/E_0, \quad \bar{\alpha}_i = \alpha_i/\alpha_0, \quad \bar{\sigma}_{ij} = \sigma_{ij}/(\alpha_0 E_0 T_0), \quad (\bar{u},\bar{v},\bar{w}) = (u,v,w)/(\alpha_0 T_0 B) \tag{17}$$

NUMERICAL CALCULATION

To illustrate the foregoing analysis, numerical parameters of heat conduction and shape are presented as follows:

$$H_a(=h_a B) = H_b(=h_b B) = 1.0, \quad \bar{T}_a = 1, \quad \bar{T}_b = 0, \quad \bar{L}_x = \bar{L}_y = 3.0,$$
$$f_a(\bar{x}) = (1-\bar{x}^2/\bar{x}_0^2)H(\bar{x}_0 - |\bar{x}|), \; g_a(\bar{x}) = (1-\bar{y}^2/\bar{y}_0^2)H(\bar{y}_0 - |\bar{y}|), \; \bar{x}_0 = 1.0, \quad \bar{y}_0 = 1.0 \tag{18}$$

where $H(x)$ is Heaviside's function. We assume that the plate is heated partially from the lower surface by surrounding media. The effects of the nonhomogeneity and orthotropy of the material on the temperature change, the displacement and the stress distributions are investigated.

We conclude that we can evaluate all stresses of the orthotropic functionally graded rectangular plate in a transient state.

Reference

[1] Vel S.S., Batra R.C., International Journal of Solids and Structures 40, 7181-7196, 2003.