CONSERVATION LAWS OF FUNCTIONALLY GRADED MATERIALS IN ELASTODYNAMICS

Shi Weichen
Department of Mechanical Engineering, Shanghai Maritime University
Pudong Avenue 1550, Shanghai 200135, China

Summary A special version of Noether's theorem for the sake of absolute invariance on invariant variational principles is applied to the Lagrangian density function for obtaining conservation laws of functionally graded materials. It is found that the mass density and Lamé's coefficients have to satisfy a set of first-order linear partial differential equations. Under the consideration of varying the volume fraction of the constituent materials, the effective mass density and Lamé's coefficients, satisfying those partial differential equations, are obtained. Four conservation laws in material space are presented. A path-independent integral, which is directly related to the dynamic energy release rate, in the moving coordinate reference attached at the tip of crack is given.

CONSERVATION LAWS IN MATERIAL SPACE

Conservation laws constitute a basic tool in the analysis of properties of solutions to any given system of partial differential equations and provide valuable information on the physical quantities pertaining to the problem under investigation, so that the conservation laws of functionally graded materials (FGMs) should be inquired.
The Lagrangian density function L and the constitutive equations of FGM are

$$L = \frac{1}{2}\rho \dot{u}_i \dot{u}_i - W(x_i, \varepsilon_{ij}) = \frac{1}{2}\rho \dot{u}_i \dot{u}_i - \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \qquad \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = C_{ijkl}u_{k,l}, \qquad C_{ijkl} = \delta_{ij}\delta_{kl}\lambda + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\mu \tag{1}$$

where W is the elastic potential; u_i, ε_{ij} and σ_{ij} are the components of displacement, strain and stress, respectively; ρ, λ and μ stand for the mass density and Lamé's coefficients, which are functions of space x_i for FGMs. By applying Lie's infinitesimal criterion on the invariance to the Lagrangian (Olver, [1]), it follows that

$$\tau = bt + \beta, \qquad \zeta_i = Bx_i + e_{ijk}\Omega_j x_k + C_i, \qquad U_i = Au_i + e_{ijk}\Omega_j u_k + e_{ijk}\omega_j x_k + \gamma_i \tag{2}$$

where β, ω_j and γ_i are arbitrary constants. Values of the constants b, B, Ω_j, C_i and A depend on satisfaction of the following equations

$$\zeta_m \frac{\partial \rho}{\partial \mathrm{x}_m} + (2A + 3B - b)\rho = 0, \qquad \zeta_m \frac{\partial \lambda}{\partial \mathrm{x}_m} + (2A + B + b)\lambda = 0, \qquad \zeta_m \frac{\partial \mu}{\partial \mathrm{x}_m} + (2A + B + b)\mu = 0 \tag{3}$$

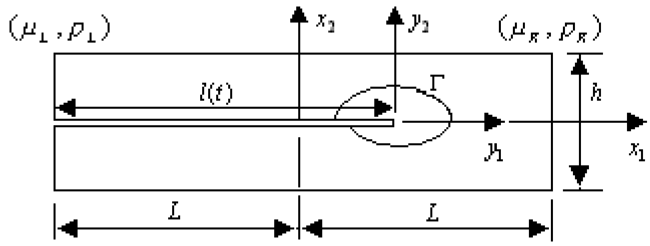

Fig.1. Propagation of a crack in functionally graded material.

FGMs are composite, microscopically non-homogenous materials, in which the material coefficients vary smoothly and continuously from one surface to the other as shown in Fig.1. This is achieved by gradually varying the volume fraction of the constituent materials. In such a way, the effective shear modulus μ may be expressed as

$$\mu = \mu_L V_L(x_1) + \mu_R V_R(x_1) \tag{4}$$

where μ_L and μ_R are the shear moduli on the left-hand and right-hand surfaces, respectively; $V_L(x_1)$ and $V_R(x_1)$ denote the volume fractions and related by

$$V_L(x_1) + V_R(x_1) = 1 \tag{5}$$

Similarly, the effective mass density ρ can be expressed in terms of volume fractions as shown above. As discussed in many other papers, Poisson's ratio ν depends weakly on position, so that it is assumed to be a constant. Under the consideration of (4) and (5), the effective shear modulus and mass density are obtained by solving equations in (3) as follows

$$\mu = \tilde{\mu}[1 + Rx_1]^N, \qquad \rho = \tilde{\rho}[1 + Rx_1]^M \tag{6}$$

where $$\tilde{\mu} = [(\mu_L^{\frac{1}{N}} + \mu_R^{\frac{1}{N}})/2]^N, \qquad \tilde{\rho} = [(\rho_L^{\frac{1}{M}} + \rho_R^{\frac{1}{M}})/2]^M, \qquad M = N[\ln(\rho_R/\rho_L)/\ln(\mu_R/\mu_L)] \tag{7}$$

$$B = RC_1, \qquad b = (M - N + 2)RC_1/2, \qquad A = -(M + N + 4)RC_1/4, \qquad R = [(\mu_R^{\frac{1}{N}} - \mu_L^{\frac{1}{N}})/(\mu_R^{\frac{1}{N}} + \mu_L^{\frac{1}{N}})]/L$$

and N is an arbitrary constant. Here, owing to V_L and V_R being functions of x_1 only, constants Ω_2 and Ω_3 within ζ_1 have been set equal to zero. The normalized values μ/μ_L and ρ/ρ_L are given in Fig.2, in which $\mu_L = 25.94$ (GPa) and $\rho_L = 2700$ (kg/m^3) for aluminum Al 6061-TO, and $\mu_R = 200.00$ (GPa) and $\rho_R = 4920\,(kg/m^3)$ for ceramic TiC. They are numerically well and would not be unrealistic for various FGMs under the consideration of varying the volume fraction.

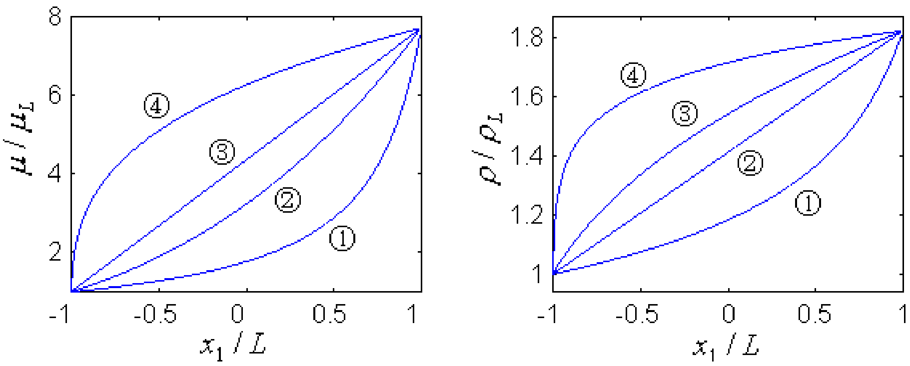

Fig.2. Effective shear modulus and mass density; ①: $N=-1$, $M=-0.2938$; ②: $N=3.4308$, $M=1$; ③: $N=1$, $M=0.2938$; ④: $N=0.3$, $M=0.0881$.

Noether's theorem [1,2] asserts the existence of conservation laws, namely

$$D_t(\tau H - \zeta_k\Sigma_k - U_kQ_k) + D_i(-\tau\dot{u}_k\sigma_{ik} + \zeta_k S_{ik} + U_k\sigma_{ik}) = 0 \tag{8}$$

where $H = W + \frac{1}{2}\rho\dot{u}_i\dot{u}_i$ is the Hamiltonian density, $\Sigma_k = -\rho\dot{u}_i u_{i,k}$ the pseudomomentum, $Q_k = \rho\dot{u}_k$ the linear momentum and $S_{ik} = (W - \frac{1}{2}\rho\dot{u}_j\dot{u}_j)\delta_{ik} - \sigma_{ji}u_{j,k}$ a dynamic version of Eshelby's energy-momentum tensor. Substituting expressions (2) into equation (8), considering the independence of C_i and Ω_1, and neglecting independent constants β, ω_j and γ_i for conservation laws in physical space, we obtain the following conservation laws in material space

(1) $C_1 \neq 0$: $\quad D_iP_i = D_tP_t \quad (9)$

where

$$P_i = S_{i1} + Rx_kS_{ik} - [1 + (1 + \frac{\ln(\rho_R/\rho_L)}{\ln(\mu_R/\mu_L)})\frac{N}{4}]Ru_k\sigma_{ik} - [1 - (1 - \frac{\ln(\rho_R/\rho_L)}{\ln(\mu_R/\mu_L)})\frac{N}{2}]Rt\dot{u}_k\sigma_{ik} \tag{10}$$

$$P_t = \Sigma_1 + Rx_k\Sigma_k - [1 + (1 + \frac{\ln(\rho_R/\rho_L)}{\ln(\mu_R/\mu_L)})\frac{N}{4}]Ru_kQ_k - [1 - (1 - \frac{\ln(\rho_R/\rho_L)}{\ln(\mu_R/\mu_L)})\frac{N}{2}]RtH$$

(2) $C_2 \neq 0$: $\quad D_iS_{i2} = D_t\Sigma_2 \quad (11)$

(3) $C_3 \neq 0$: $\quad D_iS_{i3} = D_t\Sigma_3 \quad (12)$

(4) $\Omega_1 \neq 0$: $\quad D_i(x_3S_{i2} - x_2S_{i3} + u_3\sigma_{i2} - u_2\sigma_{i3}) = D_t(x_3\Sigma_2 - x_2\Sigma_3 + u_3Q_2 - u_2Q_3) \quad (13)$

It should be emphasized that these conservation laws under the consideration of absolute invariance of the Lagrangian are non-trivial.

THE PART-INDEPENDENT INTEGRAL

By using the effective shear modulus and mass density given in (6), it can be reaffirmed that the dominant terms in the crack tip stress field are identical to those of a homogeneous material having the material properties of the FGM crack tip vicinity. It is also known that near the tip of an extending crack, field quantities obey the "transport assumption", that is, $\partial(\ )/\partial t = -\dot{l}\partial(\ )/\partial x_1$. Therefore, as shown in Fig.1, the path-independent integral emanating from the conservation law (9) in the moving coordinate reference (y_1, y_2) attached at the tip of crack can be calculated as

$$J = \lim_{\Gamma\to 0}\int_\Gamma [(P_1 + \dot{l}P_t)n_1 + P_2n_2]d\Gamma = \{1 + Rl - [1 - \frac{N}{2}(1 - \frac{\ln(\rho_R/\rho_L)}{\ln(\mu_R/\mu_L)})Rt\dot{l}]\}G \tag{14}$$

where G is the dynamic energy release rate (Freund, [3]).

References

[1] Olver, P. J.: Applications of Lie Groups to Differential Equations. Springer-Verlag, NY 1993.

[2] Noether, E.: Invariant variational problems. *Klg-Ges. Wiss. Nach. Göttingen, Math. Physik*, **Kl.**2, 235-253, 1918. English Translation: Transport Theory and Statistical Physics **1**: 183-207, 1971.

[3] Freund, L. B.: Dynamic Fracture Mechanics. Cambridge University Press, NY 1990.