ARC-LENGTH METHOD FOR EXPLICIT DYNAMIC RELAXATION

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Summary The study comprises the use of the arc length method to trace convolute load-deflection paths, typical of buckling problems, in association with dynamic relaxation technique, an explicit iterative method for the static solution of structural mechanics analyses. The work presents some new advances on the combination of these two techniques that are suitable to handle highly nonlinear limit point and snap-through problems.

DYNAMIC RELAXATION

The technique consists of finding the static solution from the transient response of a system excited through a suddenly loading. The dynamic equilibrium equations corresponding to n degrees of freedom of the system are solved for a damping coefficient close to the critical one. After some initial oscillations both velocities and accelerations tend to zero while the vector of displacements approaches to the deformed equilibrium configuration. It is based on the time domain integration of recursive eqs (1) and (2) related to the k^{th} time increment to determine the discrete displacement vector $\boldsymbol{q}$:

$$\dot{\boldsymbol{q}}^{k+1/2} = \frac{(2-c\Delta t)}{(2+c\Delta t)}\dot{\boldsymbol{q}}^{k-1/2} + \frac{2\boldsymbol{M}^{-1}\boldsymbol{r}^k\Delta t}{(2+c\Delta t)} \ , \tag{1}$$

$$\boldsymbol{q}^{k+1} = \boldsymbol{q}^k + \dot{\boldsymbol{q}}^{k+1/2}\Delta t \quad \text{and} \tag{2}$$

$$\boldsymbol{r}^k = \boldsymbol{P}^k - \boldsymbol{F}^k \ , \tag{3}$$

where $\boldsymbol{M}$ is a mass diagonal matrix (nxn), $\boldsymbol{r}$ the residual vector, c the damping coefficient and the dot ($^\bullet$) means derivative in relation to the time t. The residual force vector is the difference between the internal force vector ($\boldsymbol{F}$) and the external load vector ($\boldsymbol{P}$). When the velocity vector $\dot{\boldsymbol{q}}$ approaches to zero the static equilibrium is then assumed. The optimum c, $\boldsymbol{M}$ and Δt parameters may be obtained as proposed by Underwood [1]. A damping coefficient close to the critical one can speed up the convergence that is obtained when $\boldsymbol{r}^k \approx 0$.

ARC-LENGTH METHOD

The arc-length method was first introduced by Riks [2] and used by Wempner [3] in order to improve the incremental computations near to limit points. Ramesh and Krishnamoorthy [4] applied the method for the first time in association with the dynamic relaxation to trace load-displacement paths for buckling analyses. A constrain condition based on the norm of the total displacements vector was employed. The method presented good results for truss and beam elements, but was not able to trace the post-buckling behavior of shells, in view of an oscillating response at the unstable equilibrium path. The arc-length method is used to calculate de load increment (δp) that becomes an unknown in the unstable region of the equilibrium path. A constrain condition is added to the n equilibrium equations to obtain the solution of $n+1$ unknowns, the n degrees of freedom and an incremental load intensity factor $\delta\Lambda$. The circular constrain condition proposed by Crisfield [5] was considered suitable to be employed in the dynamic relaxation technique,

$$\left\|\delta\boldsymbol{q}^{k+1}\right\|^2 + \left(\delta\Lambda^{k+1}\right)^2 = S^2 \quad \text{or} \quad \sum_{i=1}^{n}\left(\delta q_{m,i}^{k+1}\right)^2 + \left(\delta\Lambda_m^{k+1}\right)^2 = S^2 \ , \tag{4a,b}$$

where S means the arc-length and the subscript m the load increment. From eqs. (1) and (2), the components of the incremental vector $\delta\boldsymbol{q}$ may be represented by the following equations:

$$\delta q_{m,i}^{k+1} = \beta\left(\delta q_{m,i}^{k} - \delta q_{m,i}^{k-1}\right) + \delta q_{m,i}^{k} + \frac{\alpha r_{m,i}^{k}}{m_{ii}} \ , \qquad \beta = \frac{(2-c\Delta t)}{(2+c\Delta t)} \ , \qquad \text{and} \qquad \alpha = \frac{2\Delta t^2}{(2+c\Delta t)} \ . \tag{5a,b,c}$$

When the theoretical model assumes one single parameter p to represent the applied external force, eq. (6) may be used to represent the external load as a function of a constant reference value (p_{ref}):

$$p_m^k = \Lambda_m^{k+1} p_{ref} \ . \tag{6}$$

If following forces (non-conservative loading) must be considered, the load vector $\boldsymbol{P}$ is given by

$$\boldsymbol{P}_m^k = \boldsymbol{P}_{m-1} + \delta\boldsymbol{P}_m^k = p_m^k \boldsymbol{g}_m^k \ , \tag{7}$$

where $\boldsymbol{g}$ is a vector in which the components are displacement functions, therefore

$$\delta \boldsymbol{P}_m^k = \delta p_m^k \boldsymbol{g}_m^k + p_m^k \delta \boldsymbol{g}_m^k = \delta p_m^k \boldsymbol{g}_m^k + \left(p_{m-1} + \delta p_m^k\right)\delta \boldsymbol{g}_m^k = \delta \Lambda_m^{k+1} p_{ref}\left(\boldsymbol{g}_m^k + \delta \boldsymbol{g}_m^k\right) + p_{m-1}\delta \boldsymbol{g}_m^k . \tag{8}$$

Using eqs (3), (7), (8) and (5a), the constrain condition (4b) may be transformed into the following second degree equation:

$$a_1\left(\delta \Lambda_m^{k+1}\right)^2 + a_2 \delta \Lambda_m^{k+1} + a_3 = 0 \ , \qquad \text{where} \tag{9}$$

$$a_1 = \left\|\boldsymbol{a}_m^k\right\|^2 + 1 \ , \qquad a_2 = 2\left(\boldsymbol{a}_m^k \cdot \boldsymbol{b}_m^k\right) \qquad \text{and} \qquad a_3 = \left\|\boldsymbol{b}_m^k\right\|^2 - S^2 \ , \tag{10a,b,c}$$

and the components of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are, respectively,

$$a_{\boldsymbol{m,i}}^{\boldsymbol{k}} = \frac{\alpha}{m_{ii}} p_{ref}\left(g_{m,i}^k + \delta g_{m,i}^k\right) \qquad \text{and} \qquad b_{m,i}^k = \beta\left(\delta q_{m,i}^k - \delta q_{m,i}^{k-1}\right) + \delta q_{m,i}^k - \frac{\alpha}{m_{ii}}\left(P_{\boldsymbol{m-1,i}} + p_{m-1}\delta g_{m,i}^k - F_{\boldsymbol{m,i}}^{\boldsymbol{k}}\right) . \tag{11a,b}$$

If the discriminant (Δ) of eq. (9) is positive, two real roots will be generated. The correct root must generate the minimum angle (maximum cosine) with the old solution $\delta \boldsymbol{q}_{m-1}$, to prevent the solution from 'doubling back on its tracks' [6]. Therefore, these roots must be used to obtain from eq. (5a) the new incremental displacement vectors $\delta^1 \boldsymbol{q}_{\boldsymbol{m}}^{k+1}$ and $\delta^2 \boldsymbol{q}_{\boldsymbol{m}}^{k+1}$. The cosine of the angles θ_1 e θ_2, between the 'new' and the 'old' increments are given by:

$$cos\theta_1 = \frac{\delta^1 \boldsymbol{q}_{\boldsymbol{m}}^{k+1} \cdot \delta \boldsymbol{q}_{\boldsymbol{m-1}}}{S^2} \qquad \text{and} \qquad cos\theta_2 = \frac{\delta^2 \boldsymbol{q}_{\boldsymbol{m}}^{k+1} \cdot \delta \boldsymbol{q}_{\boldsymbol{m-1}}}{S^2} \ . \tag{12a,b}$$

Unfortunately, near to limit points, the discriminant assumes positive values close to zero and complex roots may occur. To overcome this problem a shift (d) of the constraint condition may be done in the following way

$$\left\|\delta \boldsymbol{q}_{\boldsymbol{m}}^{k+1}\right\|^2 + \left(\delta \Lambda_m^{k+1} - d\right)^2 = S^2 \ . \tag{13}$$

The correct value of d must generate $\Delta \geq 0$ or

$$\Delta = 4\left(\boldsymbol{a}_m^k \cdot \boldsymbol{b}_m^k - d\right)^2 + 4\left(\left\|\boldsymbol{a}_m^k\right\|^2 + 1\right)\left(S^2 - d^2 - \left\|\boldsymbol{b}_m^k\right\|^2\right) \geq 0 \ . \tag{14}$$

The solution of eq. (14) generates two roots d_1 e d_2 , given by eq. (15). The correct value of d must lie between d_1 and d_2,

$$d_{1,2} = \left\{-\left(\boldsymbol{a}_m^k \cdot \boldsymbol{b}_m^k\right) \pm \sqrt{\left(\left\|\boldsymbol{a}_m^k\right\|^2 + 1\right)\left[\left(\boldsymbol{a}_m^k \cdot \boldsymbol{b}_m^k\right)^2 + \left\|\boldsymbol{a}_m^k\right\|^2\left(S^2 - \left\|\boldsymbol{b}_m^k\right\|^2\right)\right]}\right\} \Big/ \left\|\boldsymbol{a}_m^k\right\|^2 \ . \tag{15}$$

CONCLUSION

The method herein described has been successfully applied to simulate the quasi-static propagation of collapse in submarine pipelines [7]. It was used in combination with the finite difference discretization of the solid structure but applications with the finite element method are also possible [4]. It can be easily adapted to others explicit vector iteration methods and is especially attractive for problems with highly nonlinear geometric and material behavior.

References

[1] Underwood, P.: Dynamic Relaxation. In: Belytschko T., Hughes T.J.R.: Computational Methods for Transient Analysis. North-Holland, Amsterdam, 245-265, 1983.

[2] Riks, E.: On the Numerical Solution of Snapping Problems in the Theory of Elastic Stability. *PhD Thesis*, Dept. Aero/Astronautics, Stanford University, Stanford, 1970

[3] Wempner, G.A.: Discrete Approximations Related to Nonlinear Theories of Solids. *Int. J. Solids Struct.* **7**:1581-1599, 1971.

[4] Ramesh, G., Krishnamoorthy, C.S.: Post-Buckling Analysis of Structures by Dynamic Relaxation. *Int. J. Num. Meth. Eng.* **36**:1339-1364, 1993.

[5] Crisfield, M.A.: A Fast Incremental/Iterative Solution Procedure that Handles Snap-Through. *Computers and Structures* **13**:55-62, 1981.

[6] Crisfield, M.A.: Non-Linear Finite Element Analysis of Solids and Structures. Vol. 1: Essentials. John Wiley & Sons, NY 1991.

[7] Pasqualino, I.P., Estefen, S.F.: A Nonlinear Analysis of the Buckle Propagation Problem in Deepwater Pipelines. *Int. J. Solids Struct.* **38**:8481-8502, 2001.