TENSOR INVARIANTS AND MECHANISMS OF TRANSITION TO CHAOS IN NONHOLONOMIC DYNAMICAL SYSTEMS

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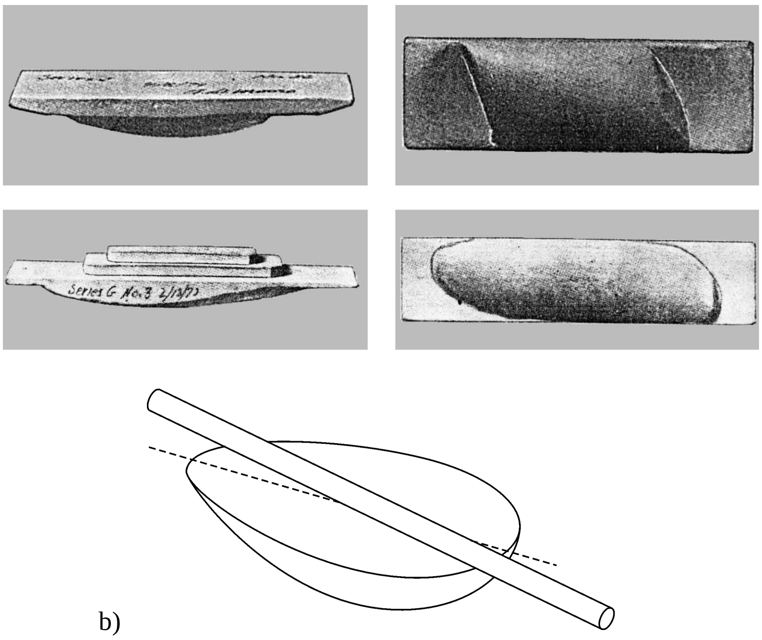
We consider nonholonomic systems that describe rolling of a rigid body on a plane and a sphere, as well as rolling of a ball on an arbitrary surface.

Any nonholonomic system has at least one nonintegrable constraint, and in our case this constraint implies that the point of contact of the body with the surface has zero velocity. Therefore, this nonholonomic model differs radically from the Hamiltonian model (with an absolutely smooth plane) and, at the same time, does not incorporate any sliding friction. Obviously, under the no-slip condition, the work done by the friction forces is zero and, therefore, the energy is conserved. Study of nonholonomic systems goes as far back as to the classical papers by S. A. Chaplygin, P. E. Appell, D. J. Korteweg, E. J. Routh, and today we can see a rapid growth of interest in such systems, due to the latest developments of the theory of dynamical systems. Nonholonomic systems show striking and often quite unusual properties. One of the most well-known examples is a so-called rattleback [1], which is a solid body with a special mass distribution, rolling on a plane (Fig. 1).



a) b)

Figure 1. The rattleback models from the paper by J. Walker.

The unusual properties of a rattleback's behavior manifest themselves in the following way. When placed on a horizontal surface and spun in a certain direction about the vertical axis, a rattleback spins freely; if, however, it is then made to rotate in the opposite direction, it soon stops rotating and starts oscillating about the horizontal axis. Some rattlebacks can reverse the direction they were spun initially.

It turns out that this and similar problems can be investigated numerically using the three-dimensional Poincaré cross-section. As such numerical investigations show, the behavior of this system can be very complex. The following theorem (proved with certain computer-assisted techniques) holds:

Theorem 1 (on complex dynamics) *Let a body be bounded with a paraboloid surface, its geometrical axes not coinciding with the axes of the inertia ellipsoid. Then the values of the system's parameters and the inertia ellipsoid are such that*

1) *there are no asymptotically stable periodic solutions in the phase space;*

2) *unstable periodic solutions exist, their stable and unstable invariant manifolds intersect transversally (i. e., a complex dynamics exists);*

3) *there is a domain in the phase space, and a phase flow on its boundaries is inward (i. e., an attractive set exists);*

4) *the Lyapunov exponent of a typical orbit is non-zero.*

Therefore, there is a strange attractor in the phase space.

Besides, we show, using numerical and analytical methods, that these classical problems (rolling of a body on a plane and on a sphere and rolling of a ball on an arbitrary surface) include various new cases when tensor invariants [2] (additional integrals and the invariant measure) exist. And there is a certain hierarchy of possible types of behavior of nonholonomic systems. The extreme cases of this hierarchy are, on one hand, quite integrable systems (according to the Euler–Jacobi theorem), which can have the full set of first integrals and the invariant measure that behave quite regularly, and on the other hand, systems that have neither integrals, nor invariant measure. The latter display chaotic behavior, which can be either Hamiltonian or dissipative (a rattleback is a good example of such a system).

There is one more new mechanical effect discovered from the numerical experiments. We consider the motion of an axisymmetric ellipsoid with special mass distribution, so that the axes of the inertia ellipsoid are rotated, with respect to its geometrical axes, about the axis perpendicular to the ellipsoid's symmetry axis Oz (Fig. 2). Spun about this axis, the body at first shows reversion, like a rattleback, i. e. only the direction of the motion is changed. But this rotation is unstable either, and the body would gradually turn over, like a Chinese top, i. e. its initially upper part would move down, and vice versa.

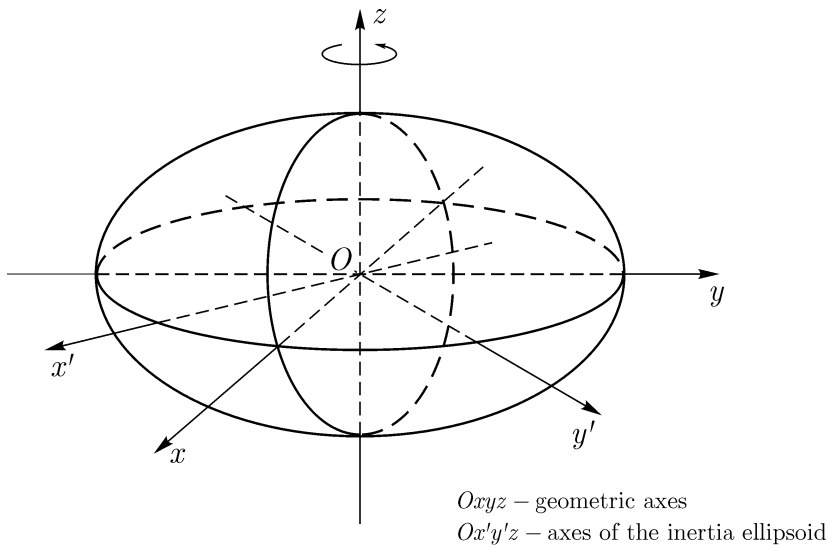


Figure 2.

References

[1] *Walker J.* The mysterious «rattleback»: a stone spins in one direction and then reverses. Scientific American, 1979, No. 10, pp. 144–149.

[2] *Kozlov V. V.* On the integration theory of equations of nonholonomic mechanics. Reg.&Chaot. Dyn. 2002, V. 7, No. 2, pp. 161–176 (in Russian).