ANALYSIS OF EVOLVING DEFORMATION MICROSTRUCTURES IN INSTABLE INELASTIC SOLIDS BASED ON ENERGY RELAXATION METHODS

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Summary An incremental variational formulation for the constitutive response of dissipative materials is applied to the stability analysis of inelastic solids. The material stability is governed by weak convexity properties of an incremental stress potential obtained from a local minimization problem. As a result of instability deformation microstructures develop and are resolved by energy relaxation methods. We develop relaxation analyses in terms of rank–one convexifications and make comparisons of results obtained from different orders of lamination as well as different kinematic degrees of freedom of the laminates.

Introduction

In problems of finite inelasticity such as in single–slip plasticity material instabilities may occur due to geometric effects. These instabilities then cause the development of fine scale microstructures. The goal of this lecture is to resolve the development of these deformation microstructures based on an incremental energy minimization method. To this end we define an incremental stress potential function depending on the current deformation $\mathbf{F}_{n+1} := \mathbf{F}(t_{n+1})$ at time t_{n+1} which determines the stresses at t_{n+1} by quasi–hyperelastic function evaluation $\mathbf{F}_{n+1} = \partial_{\mathbf{F}} W(\mathbf{F}_{n+1})$. Here W is defined by a minimization problem proposed in [2]

$$W(\mathbf{F}_{n+1}) = \inf_{\boldsymbol{\mathcal{I}}\in\mathcal{G}} \int_{t_n}^{t_{n+1}} [\,\dot{\psi} + \phi\,]\, dt \quad \text{with} \quad \boldsymbol{\mathcal{I}}(t_n) = \boldsymbol{\mathcal{I}}_n \tag{1}$$

where ψ and φ denote the free energy and dissipation functions, respectively. For a prescribed deformation, this problem defines the incremental stress potential function W as a minimum of the generalized work $\int_{t_n}^{t_{n+1}} [\,\dot{\psi} + \phi\,]\, dt$ with respect to internal variables $\boldsymbol{\mathcal{I}}$ done on the material in the time increment under consideration. The minimization problem is approximated by a finite–step–sized algorithm and serves as the fundamental basis of the subsequent stability analysis.

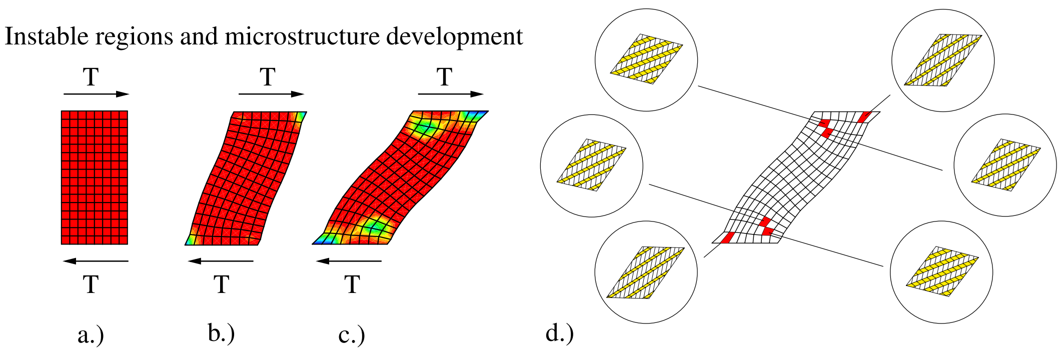

Figure 1. Rectangular specimen in shear. a - c.) Evolution of instable regions, d.) visualization of deformation microstructures.

Material Stability Analysis of Standard Dissipative Solids

As already pointed out in [1, 3, 4, 5], a key advantage of the above mentioned variational formulation is the opportunity to analyze the incremental stability of inelastic solids in terms of terminologies used in finite elasticity. The existence of the constitutive minimization problem (1) allows the introduction of an incremental minimization formulation of the boundary value problem of finite inelasticity for standard dissipative solids in a typical time increment. We consider a displacement functional $I(\boldsymbol{\varphi}_{n+1}) = \int_{\mathcal{B}} W(\mathbf{F}_{n+1}) dV - [\Pi_{ext}(\boldsymbol{\varphi_{n+1}}) + \Pi_{ext}(\boldsymbol{\varphi_n})]$ of the current deformation field $\boldsymbol{\varphi}_{n+1}$ with contribution Π coming from body forces and surface tractions. The current deformation map of inelastic standard dissipative materials can then be determined by a principle of minimum incremental energy $I(\boldsymbol{\varphi^*}_{n+1}) = \inf_{\boldsymbol{\varphi_{n+1}}} I(\boldsymbol{\varphi}_{n+1})$ subject to essential boundary conditions. This minimization problem governs the response of inelastic solid in the finite increment $[t_n, t_{n+1}]$ in a structure identical to the principle of minimum potential energy in finite elasticity. Extending the results of the existence theory in finite elasticity to the incremental response of standard dissipative solids in the finite step $[t_n, t_{n+1}]$, we consider *quasiconvexity* of the incremental potential W as the fundamental criterion for the incremental material stability of the inelastic solid. W is said to be *quasiconvex* at $\mathbf{F}_{n+1}$ if the condition

$$\frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla \mathbf{w}(\mathbf{y})) dV \geq W(\mathbf{F}_{n+1}) \tag{2}$$

with $\mathbf{y} \in \mathcal{D}$ is satisfied for all $\mathbf{w}$ having $\mathbf{w} = \mathbf{0}$ on $\partial\mathcal{D}$. Here, $\mathcal{D} \in \mathcal{R}^3$ is an arbitrarily chosen part of the inelastic solid and $\partial\mathcal{D}$ is its boundary. The quasiconvexity condition states for all fluctuations $\mathbf{w}$ on $\mathcal{D}$ with support on $\partial\mathcal{D}$, the homogeneous deformation $\mathbf{F}_{n+1}$ provides an absolute minimizer of the incremental potential in $\mathcal{D}$. This condition extends results obtained by [7] for continuous rate formulation to a finite–step–sized incremental setting.

Relaxation of Non–Convex Response and Development of Microstructures

If the incremental potential W becomes non–quasiconvex, then material instability is detected at a point of the solid and accordingly the existence of the solutions is not ensured. In order to overcome this problem we consider the *relaxed energy functional* $I_Q(\boldsymbol{\varphi}_{n+1}) = \int_{\mathcal{B}} W_Q(\boldsymbol{\varphi}_{n+1})dV - [\Pi_{ext}(\boldsymbol{\varphi}_{n+1}) - \Pi_{ext}(\boldsymbol{\varphi}_n)]$ where the relaxed potential is obtained by replacing the non–convex integrand W by its *quasiconvex* envelope W_Q. The quasiconvexified incremental stress potential is defined by the minimization problem with respect to the microscopic fluctuation field

$$W_Q(\mathbf{F}_{n+1}) = \inf_{\mathbf{w}\in\mathcal{W}_0} \frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla\mathbf{w}(\mathbf{y}))dV. \tag{3}$$

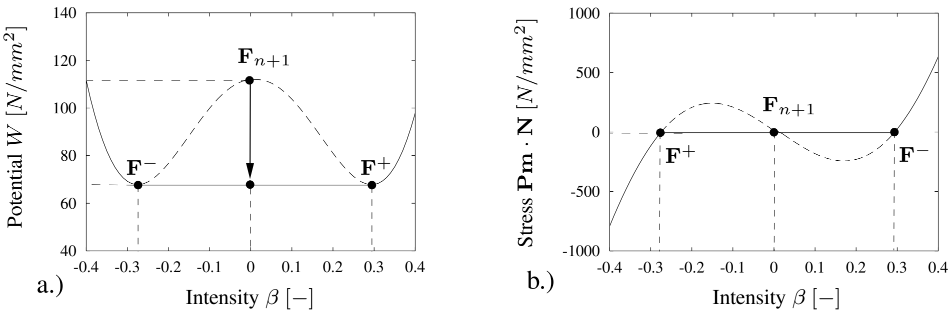

Figure 2. Simple shear test. Details of the rank–one convexification. β parameterizes the intensity of the laminate $\mathbf{F}^{\pm} = \mathbf{F}_{n+1} + \beta^{\pm}\mathbf{m}\otimes\mathbf{N}$. a.) At $\mathbf{F}_{n+1}$ the potential is not rank–one convex (dashed line). $\mathbf{F}_{n+1}$ decomposes into micro–phases $\mathbf{F}^{\pm}$ (solid line). b.) The relaxed stress-strain relation characterizes a snap–through Maxwell–line behavior between the micro–phases $\mathbf{F}^{\pm}$.

The current deformation field of the standard dissipative solid is determined by the relaxed incremental variational problem $I_Q(\boldsymbol{\varphi}^*{}_{n+1}) = \inf_{\boldsymbol{\varphi}_{n+1}} I_Q(\boldsymbol{\varphi}_{n+1})$. The relaxed problem is considered to be a well–posed problem as close as possible the original instable problem. In Figure 2 the non–convex character of the incremental potential and corresponding stress response are visualized for the model problem of single slip plasticity. Since the quasiconvexity is hard to handle in practice, we concentrate on the slightly weaker rank–one–convexification W_R of the potential W for different orders by sequential laminations. The simplest case is a first–order rank–one–convex envelope W_{R_1} of W defined by

$$W_{R_1}(\mathbf{F}_{n+1}) = \inf_{\xi,d,\mathbf{m},\mathbf{N}} \{\xi W(\mathbf{F}^+) + (1-\xi)W(\mathbf{F}^-)\} \tag{4}$$

where $\mathbf{F}^+ = \mathbf{F} + (1-\xi)d\mathbf{m}\otimes\mathbf{N}$, $\mathbf{F}^- = \mathbf{F} - \xi d\mathbf{m}\otimes\mathbf{N}$, ξ and $1-\xi$ represent the rank–one connected deformation phases and the corresponding volume fractions. Higher order relaxed response is obtained by further minimizing of each phase by first–order laminates. The lecture points out mathematical relaxation approaches and discusses the physical relevance of the developing microstructures. This is done by considering

- a first order rank–one relaxation with keeping all the parameters free [5], i.e. minimization of energy is obtained with respect to the set $\mathbf{q} = [\xi, d, \mathbf{m}, \mathbf{N}]$. Figure 1 shows for first–order laminate microstructures developed in a shear test,
- a sequential lamination with fixed laminate orientation motivated by physical observations as conceptually investigated in [6, 3, 4].

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