EFFECTIVE SOLUTION FOR FINITE ELEMENT PROBLEMS WITH NONLINEAR CONSTRAINTS

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Introduction

Very often solutions of mechanical problems based on the Finite Element Method are accompanied by a system of nonlinear equations. FEM creates a large number of linear equations but the requirements of laws of physic, optimization techniques, geometrical non-linearity's, adaptive modeling result in nonlinear equations. In this case the whole system is considered nonlinear and is solved using methods for the solution of non-linear equations. The Newton-Raphson iterations is the method most commonly used. This method needs the calculation of first derivatives and the Jacobian matrix for the system. The solution is obtained by means of consecutive iterations. If the functions are differentiable with respect to the unknown variables and behave well, it is possible to find the solution in a reasonable number of iterations. However, this method needs the initial guess for all the variables taken sufficiently close to the simultaneous roots of the nonlinear system. The approach is not effective if the number of equations is large. There are problems with the convergence to correct the solution and problems with the initial guesses for the variables.

Method

This paper presents a new method for the solution of a system of $m+n$ nonlinear equations when the system of equations can be presented as two groups of equations. The first group of m equations is linear with respect to the selected m variables; the second group of n equations is nonlinear. The solution for the first group does not require any iterative procedures and can be found by means of any method for the system of linear equations. The method uses iterations only for the nonlinear part and needs therefore much fewer number of initial guesses as compared to those needed in Newton-Raphson method.
The general system of equations can be presented in the following form:

$$\mathbf{f}(x,t) = 0,$$

$$\boldsymbol{\varphi}(x,t) = 0,$$

where the system f_i is linear with respect to the variables x_i with the assumption that the values of the variables t_i are known. Equations φ_n are non-linear with respect to the variables x_i and $\mathbf{t_i}$. Suppose that the vector t is the initial guess solution to the nonlinear variables of the system of the equations. Similarly, the vector x is the vector of initial solution for the linear part of the system equations based on t. The vector x can be found by means of any method for the system of linear equations. Let $x+\Delta x$ and $t+\Delta t$ be a better approximate solution.
Representing the functions f and $\boldsymbol{\varphi}$ by Taylor expansions, we have

$$\boldsymbol{\varphi}(x,t) + \frac{\partial \boldsymbol{\varphi}}{\partial x}\Delta x + \frac{\partial \boldsymbol{\varphi}}{\partial t}\Delta t = 0, \qquad \mathbf{f}(x,t) + \frac{\partial \mathbf{f}}{\partial x}\Delta x + \frac{\partial \mathbf{f}}{\partial t}\Delta t = 0 . \tag{1}$$

Solving equations (1) and with respect to Δx and Δt gives:

$$\Delta \mathbf{t} = -\left[I - (\frac{\partial \boldsymbol{\varphi}}{\partial t})^{-1}(\frac{\partial \boldsymbol{\varphi}}{\partial x})(\frac{\partial \mathbf{f}}{\partial x})^{-1}(\frac{\partial \mathbf{f}}{\partial t}) \right]^{-1} \left[\frac{\partial \boldsymbol{\varphi}}{\partial t} \right]^{-1} [\boldsymbol{\varphi}(x,t)], \tag{2}$$

and $$\Delta \mathbf{x} = -\left[\frac{\partial \mathbf{f}}{\partial x}\right]^{-1}\left[\mathbf{f}(x,t) + \frac{\partial \mathbf{f}}{\partial t}\Delta t\right]. \quad (3)$$

The new values of **t** and **x** are calculated as $\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta \mathbf{x}_i, \quad \mathbf{t}_{i+1} = \mathbf{t}_i + \Delta \mathbf{t}_i$. Below a simple example is presented to explain the method. Let us consider the following system of algebraic equations. The variables to be found are *x*, *y*, *z* and *t*.

$$\begin{aligned} &tx + 2y - z = 21, \\ &-2x + 3y - z = 0, \\ &2x - y + 5z = 26 \\ &tx^2 - 2xy + y^2 - tz^2 = -213. \end{aligned} \quad (4)$$

The first and last equations are clearly nonlinear. However, following the procedure explained in the previous section, if we consider *t* as the variable to be found by iterations, the first three equations will be linear in terms of *x*, *y* and *z* and can be solved for any given ***t***.

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t & 2 & -1 \\ 2 & -1 & 5 \\ -2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 21 \\ 26 \\ 0 \end{bmatrix}. \quad (5)$$

The table presented below provides the record of the number of iterations and corresponding residue of ϕ_1 for different initial guesses t_1. The method found three independent solutions for x, y, z, and t. The most important fact is that it is almost independent on the initial value of the variable **t** and converges quickly to the solution. Three independent solutions were found. The number of iterations is very low, between 1 and 8.

Initial guess for *t*	***x***	***y***	***z***	***t***	No. of iterations	ϕ_1 *(x,y,z,t)* error
0.5	15.6300	10.7900	1.1036	0.0331	6	-0.00004
1	-10.6594	-4.2339	8.6170	-3.5729	8	-0.00013
2	2.0000	3.0000	5.0000	10.0000	4	0.00170.003
5	2.0000	3.0000	5.0000	10.0000	3	0.001
7	2.0000	3.0000	5.0000	10.0000	3	0.000002
9	2.0001	3.0001	5.0000	9.9993	2	0.0232
9.9	2.0000	3.0000	5.0000	10.0000	2	0.0003
10	2.0000	3.0000	5.0000	10.0000	1	0
50	2.0000	3.0000	5.0000	9.9999	3	0.0034
100	2.0000	3.0000	5.0000	9.9998	3	0.0068
1000	2.0001	3.0000	5.0000	9.9996	3	0.0117
10000	2.0001	3.0000	5.0000	9.9996	3	0.0124

Final Remarks and Conclusions

The method was tested on numerous examples with large number of equations and nonlinear unknowns. It has been found that the algorithm is very effective in all cases and is able to find many solutions for **t** with very high accuracy**.** It is especially suitable for the solution of identification problems, adaptive modeling, Adaptive Matrix Filter [2], where the identification and modeling is based on the large number of experimental data and model equations. It eliminates the problems with the convergence and initial guess for the all the variables. The presented approach makes possible to find the solution for many problems that were not solved due to the lack of convergence and effectiveness of the other methods. It gives often more effective solution than the method of neural networks.

References

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