Multiscale Methods in Turbulence

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Extended Summary

Variational multiscale concepts for Large Eddy Simulation (LES) were introduced in Hughes et al. [7]. The basic idea was to use variational projections in place of the traditional filtered equations and to focus modeling on fine-scale equations rather than coarse-scale equations. Avoidance of filters eliminates many difficulties associated with the traditional approach, namely, inhomogeneous non-commutative filters necessary for wall-bounded flows, use of complex filtered quantities in compressible flows, the closure problem, etc. In addition, modeling confined to the fine-scale equations retains numerical consistency in the coarse-scale equations and thus permits full rate-of-convergence of the underlying numerical method in contrast with the usual approach which limits convergence rate due to artificial viscosity effects in the fully resolved scales ($O(h^{4/3})$ in the case of Smagorinsky-type models). Initial versions of the variational mulstiscale method focused on dividing resolved scales into coarse and fine designations, and eddy viscosities, inspired by traditional models, were *only* included in the fine scale equations and acted *only* on the fine scales. This version was studied in [8, 9, 16] and found to work very well on homogeneous isotropic flows and fully-developed equilibrium and non-equilibrium turbulent channel flows. Static eddy viscosity models were employed in these studies but superior results were subsequently obtained through the use of dynamic models, as reported in Holmen et al. [4] and Hughes et al. [11]. Good numerical results were obtained with the static approach by other of investigators, namely, Collis [2], Jeanmart and Winckelmans [14] and Ramakrishnan and Collis [18, 19, 20, 21]. Particular mention should be made of the work of Farhat and Koobus [3], and Koobus and Farhat [15], who have implemented this procedure in an unstructured mesh, finite volume, compressible flow code, and applied it very successfully to a number of complex test cases and industrial flows. We believe that this initial version of the variational multiscale concept has already demonstrated its viability and practical utility and is, at the very least, competitive with traditional LES turbulence modeling approaches.

Nevertheless, there is still significant room for improvement. The use of traditional eddy viscosities to represent fine-scale dissipation is an inefficient mechanism. Employing an eddy viscosity in the resolved fine scales to represent turbulent dissipation introduces a consistency error which results in the resolved fine scales being "sacrificed" to retain full consistency in the coarse scales. (In our opinion, this is still better than the traditional approach in which consistency in *all* resolved scales is sacrificed to represent turbulent dissipation.) This procedure is felt to be "inefficient" because approximately 7/8 of the resolved scales are typically ascribed to the fine scales. Another shortcoming noted for the initial version of the variational multiscale method is too small an energy transfer to unresolved modes when the discretization is very coarse (see, e.g., Hughes et al. [11]). This phenomenon is also noted for some traditional models, such as the dynamic Smagorinsky model [11], but seems to be somewhat more pronounced for the multiscale version of the dynamic model. The objectives of recent multiscale work have been to capture all scales consistently and to avoid use of eddy viscosities altogether. This holds the promise of much more accurate and efficient LES procedures. In this work, we describe a new variational multiscale formulation which makes considerable progress toward these goals. In what follows, all resolved scales are viewed as coarse scales, which obviates the issue of inefficiency *ab initio*.

We begin by taking the view that the decomposition into coarse and fine scales is exact. For example, in the spectral case, the coarse-scale space consists of all Fourier modes beneath some cut-off wave number and the fine-scale space consists of all remaining Fourier modes. Consequently, the coarse-scale space has finite dimension whereas the fine-scale space is infinite dimensional. The derivation of the coarse- and fine-scale equations proceeds, first, by substituting the split of the exact solution into coarse and fine scales into the Navier-Stokes equations, then, second, by projecting this equation into the coarse- and fine-scale subspaces. The projection into coarse scales is a finite dimensional system for the coarse-scale component of the solution, which depends parametrically on the fine-scale component. In the spectral case, in addition to the usual terms involving the coarse-scale component, only the cross-stress and Reynolds-stress terms involve the fine-scale component. In the case of non-orthogonal bases, even the linear terms give rise to coupling between coarse and fine scales. The coarse-scale component plays an analogous role to the filtered field in the classical approach, but has the advantage of avoiding all problems associated with homogeneity, commutativity, walls, compressibility, etc. The projection into fine scales is an infinite-dimensional system for the fine-scale component of the solution which depends parametrically on the coarse-scale component. We also assume the cut-off wave number is sufficiently large that the philosophy of LES is appropriate. For example, if there is a well-defined inertial sub-range, then we assume the cut-off wave number resides somewhere within it. This assumption enables us to further assume that the energy content in the fine scales is small compared with the coarse scales. This turns out to be *crucial* in our efforts to analytically represent the solution of the fine-scale equations. The strategy is to obtain approximate analytical expressions for the fine scales then substitute them into the coarse-scale equations which are, in turn, solved numerically. If the scale decomposition is performed in space and time, the *only* approximation in the procedure is the representation of the fine-scale solution. To provide a framework for the fine-scale approximation, we assume an infinite perturbation series expansion to treat the fine-scale nonlinear term in the fine-scale equation. By virtue of the smallness of the fine scales, this expansion is expected to converge rapidly under the circumstances described in many cases of practical interest. The remaining part of the fine-scale Navier-Stokes system is the *linearized* operator which is formally inverted through the use of a Green's function. The combination of a perturbation series and Green's function provides an exact formal solution of the fine-scale Navier-Stokes equations. The driving force in these equations is the Navier-Stokes system residual computed from the coarse scales. This expresses the intuitively obvious fact that if the coarse scales constitute a good approximation to the solution of the problem, the coarse-scale residual will be small and the resulting fine-scale solution will be small as well. This is the case we have in mind and it provides a rational basis for assuming the perturbation series converges rapidly. Note that one cannot use such an argument on the original problem because in this case the perturbation series would almost definitely fail to converge. (If we could have used this argument, we would have solved the Navier-Stokes equations analytically! Unfortunately, it does not work.) The formal solution of the fine-scale equations suggests various approximations may be employed in practical problem solving. We are tempted to use the word "modeling" because approximate analytical representations of the fine scales constitute the only approximation and hence may be thought of as the "modeling" component of the present approach but we want to emphasize that it is very different from classical modeling ideas which are dominated by the *addition* of *ad hoc* eddy viscosities. We will present numerical results that demonstrate these eddy-viscosity terms are unnecessary in the present circumstances. There are two aspects to the approximation of the fine scales: 1) Approximation of the Green's function for the linearized Navier-Stokes system; and 2) approximation of the nonlinearities represented by the perturbation series. The first and obvious thought for the latter aspect, nonlinearity, is to simply truncate the perturbation series. This idea is investigated, as well as another promising idea, in conjunction with some simple approximations of the Green's function. It turns out there is considerable experience in local scaling approximations of the Green's function based on the theory of *stabilized methods* [5, 6, 10]. The Green's function is typically approximated by locally defined algebraic operators (i.e., the "τ's" of stabilized methods) multiplied by local values of the coarse-scale residual. With this approximation of the solution of the linearized operator, nonlinearities can be easily accounted for in perturbation series fashion. Another approach that accounts for nonlinearities in the fine-scale equations is to introduce a nonlinear algebraic scaling of the Navier-Stokes equations. The resulting local nonlinear algebraic system can be analytically solved. It possesses the reasonable analytical property that if the coarse-scale residual is small, it converges to the linearized solution.

These newer variational multiscale ideas, and the older variants, were implemented in a finite volume code that has enjoyed widespread use in turbulence simulations (see Pierce and Moin [17]). Following along the lines of Jacobs and Durbin [12, 13], who performed DNS investigations of bypass transition of a boundary layer, we examine this difficult problem from the point of view of the variational multiscale and classical LES. Our aim was to solve this problem as an LES and demonstrate the efficacy of the new ideas in the process.

In our work we endeavor to show the effectiveness, or deficiencies, of LES approaches by studying them over a range of resolutions, from coarse to fine. In our studies of bypass transition we went as far as DNS in the fine-scale end of the spectrum, and 1/8 DNS resolution in each spatial direction. The coarsest LES mesh then involves about 1/512 the number of equations as the DNS mesh and approximately 1/4,096 of the computational effort. We found, independent of the LES method, that in order to accurately simulate bypass transition, the decay of input homogeneous, isotropic, free-stream turbulence must be the same for all meshes. A procedure was developed in which we were able to simulate consistent energy decay with distance of the free-stream turbulence across the range of meshes considered. We then compared the methods to represent bypass transition. We found that the "1/8 DNS mesh" was incapable of representing either the laminar region of the boundary layer or the free-stream turbulence evolution due to too few points in the wall normal direction. We found that all methods gave essentially the same solution at the DNS level, whereas only the new variational multiscale formulation was able to attain relatively mesh independent solutions without parameter adjustment for the 1/4, 1/2 and full DNS mesh cases. The 1/4 DNS involves 1/64 the number of mesh points as the DNS case and 1/512 the computational effort. Comparison is made with classical LES procedures, such as the dynamic Smagorinsky model, which tends in this instance to exhibit significant sensitivity to the filter-width ratio, and previous versions of the variational multiscale method in which a fine-scale dynamic Smagorinsky model is employed. Comparison is also made with some classical stabilized methods, such as SUPG (see Brooks and Hughes [1]). We conclude that the newest method is superior to all previous methods and offers a promising new path for turbulence research in LES. However, it obviously needs testing on a wider variety of flows and implementation in a variety of numerical frameworks, such as, spectral, finite difference and finite element, before definitive conclusions can be drawn. In our experience, the particular numerical discretization method has an enormous impact on the results, and its influence is often underestimated by practitioners evaluating models.

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