MECHANICS OF RUBBERLIKE SOLIDS

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Summary In this lecture we discuss (i) the large deformation stress-strain response of rubberlike solids based on experimental observations, (ii) the mathematical modelling of this behaviour through its phenomenological treatment using elasticity theory and extensions of the theory to account for inelastic responses such as the Mullins effect and hysteretic stress-strain cycling, (iii) an introduction to the analysis of magnetoelastic effects in magneto-sensitive elastomers.

EXTENDED SUMMARY

Description of experimental results

The lecture begins with an overview of the large deformation stress-strain response of rubberlike solids based on experimental observations, in particular of vulcanized natural rubber. First, experimental results that characterize the elastic behaviour of rubber are described. This is followed by illustrations of how the behaviour departs from the purely elastic. When subjected to cyclic loading many elastomers (and also biological soft tissues) exhibit a *stress softening* phenomenon widely known as the *Mullins effect* (Mullins [1]). The Mullins effect is associated mainly with a significant reduction in the stress at a given level of strain on unloading as compared with the stress on initial loading on the first and successive cycles in stress-strain cyclic tests of fixed amplitude. An example of this behaviour is shown in Figure 1.

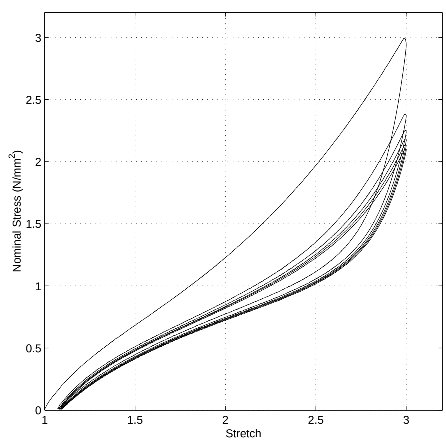

Figure 1. The Mullins effect (first loading-unloading cycle) and subsequent stress-stretch cycles for simple tension of a particle-reinforced specimen of rubber filled with 20 phr of carbon black up to a maximum stretch $\lambda = 3$ (after Dorfmann and Ogden [2]).

This is just one of the various inelastic effects that are evident in materials such as rubbers hardened by the inclusion of fillers. Other inelastic effects include frequency-dependent response and hysteresis. The Mullins effect is closely related to the fatigue of elastomeric parts used in engineering applications. A detailed qualitative and quantitative understanding of the Mullins effect is thus a necessary step towards the scientific evaluation of the *life* of a rubber product.

In this talk we examine stress softening associated with the Mullins effect, and the different degrees of stress softening for different rubbers are highlighted. Other inelastic effects such as hysteretic stress-strain cycling following pre-conditioning of the material (to remove the Mullins effect) are also described.

Theoretical approaches

Many phenomenological theories aimed at modelling the Mullins effect have been proposed in the literature. Some of these are based on the two-phase micro-structural model introduced by Mullins and Tobin [3] and developed more recently by, for example, Johnson and Beatty [4]. Another class of models is based on the introduction of a damage (or stress-softening) variable to describe the internal damage in the bulk material associated with the Mullins effect (see, for example, Gurtin and Francis [5], De Souza Neto et al. [6], Beatty and Krishnaswamy [7] and Zúñiga and Beatty [8] and references cited therein). Ogden and Roxburgh [9, 10] have developed a model based on a theory of *pseudo-elasticity* in which a pseudo-energy function is introduced as in standard nonlinear elasticity theory except that an additional scalar variable (the damage variable) is incorporated. Many of these models share the common feature that the virgin material response is determined by a standard strain-energy function or a stress constitutive function for a perfectly elastic isotropic material. The latter theory is examined in detail to illustrate how the theory can reproduce the main qualitative features of the phenomena in question. A connection between this theory and that of materials with multiple natural configurations due to Rajagopal and Wineman [11] is also noted.

A model of the Mullins effect based on micro-mechanical continuum damage theory was proposed by Govindjee and Simo [12]. The constitutive equation for this model involves the integration of the inverse Langevin function over a history-dependent domain in the phase space of the rubber network. Recently Marckmann et al. [13] have proposed another model using the inverse Langevin function in the description of network alteration associated with the Mullins effect. The non-Gaussian statistics allow one to take into account the limiting stretch of the chains composing the polymeric network. In the Marckmann et al. [13] model the Mullins effect is explained by reference to an alteration of the average chain length driven by the previous maximum stretch in a given deformation. The network alteration idea has been investigated recently by Horgan et al. [14] on the basis of the phenomenological model proposed by Gent [15] for rubberlike elasticity. This is a very simple strain-energy function that accounts for the limiting chain extensibility of rubberlike polymers (for further discussion of the model we refer to Horgan and Saccomandi [16, 17, 18]). The use of the Gent model allows one to bypass the computational problems associated with the inverse Langevin function and to provide a simpler and deeper interpretation of the network alteration phenomenon.

An important feature of the effect of stress softening and any associated residual strains is the change in material symmetry generated. This has received very little attention in the literature thus far, but here we examine some aspects of this briefly following the recent discussion in Horgan et al. [14]. An extensive review of different experimental and theoretical aspects of rubber mechanics and thermomechanics, from the molecular to the phenomenological level, is provided in the volume edited by Saccomandi and Ogden [19], which contains many pointers to the literature.

Magneto-sensitive elastomers

The final part of the talk is motivated by recent renewed interest in the subject of electromagnetic continua generated by the development of so-called ‘smart’ materials used, for example, in devices for controlling the damping characteristics of vibration absorbers. In particular, we are concerned here with elastomers containing a distribution of ferrous particles embedded within their bulk so that they respond to the application of magnetic fields by changing significantly their stress-strain behaviour. We outline recent theoretical work concerning the coupling of mechanical and magnetic effects in these so-called *magneto-sensitive (MS) elastomers*. Some analysis is contained in the papers by Dorfmann and Ogden [20, 21], but in this talk we touch on a refined version of the theory that enables boundary-value problems to be formulated in a very simple and elegant way [22]. References to the (limited) literature are given in these papers. The theory is applied to some simple prototype boundary-value problems of practical interest, notably the axial or azimuthal shear and the combined extension and torsion of MS material contained within a circular cylindrical tube in the presence of either an axial magnetic field (uniform) or an azimuthal field, the latter generated by an axial steady current within the core of the tube. The influence of the magnetic field strength on the various shear stiffnesses of the material is evaluated.

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