PROBABILITY PHENOMENA IN PERTURBED DYNAMICAL SYSTEMS

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Summary We discuss probability phenomena associated with passages through separatices and passages through resonances in perturbed dynamical systems. The theory which describes these phenomena has applications in different problems including problems of capture of satellites into resonances, of acceleration of charged particles, of chaotic advection of impurities.

INTRODUCTION

If in a deterministic dynamical system a small variation of initial data produces a large variation of dynamics, then the behavior of this system can be treated as a random one. This non-rigorous assertion known as a principle "small causes and big effects" is in the basis of the theory of deterministic chaos. Remarkably, such quasi-random behavior exists also in systems that differ by an arbitrarily small perturbations from systems with very simple (periodic, quasi-periodic) dynamics. Different types of the perturbed dynamics have certain probabilities. Analysis of long-term dynamics leads to random walk problems. In the talk we discuss probability phenomena associated with passages through separatices and passages through resonances in perturbed dynamical systems.

PASSAGES THROUGH SEPARATRICES

The problems with passages through separatrices are similar to those of motion of a material point in one dimension in a double-well potential in presence of a small, of order ε, friction (example by V.I.Arnold, 1963). The problem without friction is the unperturbed one. Consider motion of points whose initial energy is higher than energy of the potential hump separating the wells. Under the action of the friction almost everyone of these points after some time (typically, of order $1/\varepsilon$) will be trapped into one of the wells. But the initial data for trapping into different wells are mixed. Small, of order ε, change of initial conditions, can change the result of evolution. So, in the limit as $\varepsilon \to 0$ the deterministic approach to the problem fails. However, it is possible to treat trapping into one or another well as random events and calculate probabilities of these events. Probabilistic approach to problems of such kind was suggested by I.M.Lifshitz, A.A.Slutzkin, V.M.Nabutovskii (1961).

Quite general situation in which similar phenomena arise can be described as follows. The unperturbed system is 1 DOF Hamiltonian system depending on parameters. The phase portrait of this system is divided into several regions ("cells") by separatrices. Small, of order ε, perturbation (small non-conservative forces, slow change of parameters) is imposed on this system. This perturbation produces an evolution. In the course of this evolution a phase point of the perturbed system crosses the separatrix of unperturbed system and passes from one cell into another. The dynamics of perturbed system can be approximately described by a version of the averaging method that takes into account that the trapping into different cells occurs with certain probabilities. In the talk the estimates of accuracy of such averaging method are presented.

The case when the perturbation is just a slow variation of parameters in time or, more generally, the perturbed system is slow-fast Hamiltonian system, is of special interest. In this case the "action" variable of the unperturbed system is the first integral of the averaged system and, therefore, approximate first integral of the perturbed system (i.e. adiabatic invariant) for motion far from separatrices. We discuss asymptotic formulas for jump of adiabatic invariant at separatrix crossings (A.V.Timofeev, 1978, J.Cary, D.Escande, J.Tennison, 1986, A.I.Neishtadt, 1986, 1987). This jump should be treated as a random value. Accumulation of results of these jumps after many passages through separatrices leads to destruction of adiabatic invariance. This phenomenon play important role, in particular, in dynamics of asteroids near the resonance (J.Wisdom, 1985), and of charged particles in the Earth magnetotail (J.Buchner, L. Zelenyi, 1989).

Similar phenomena of jumps of adiabatic invariants at separatrix and destruction of adiabatic invariance take place in volume-preserving systems (K.Bajer, H.K.Moffatt, 1990, D.L.Vainshtein, A.A.Vasiliev, A.I.Neishtadt, 1996).

PASSAGES THROUGH RESONANCES

Consider a system which differs from an integrable multi-frequency system by a small, of order , perturbation. Motion in such a system can be approximately represented as slowly evolving unperturbed one. For description of this evolution the classical averaging method prescribes to average differential equations of a perturbed motion over the phases of unperturbed oscillations. In the course of the evolution the frequencies of motion are changing slowly and at some time moments they become resonant, i.e. linearly rationally dependent. Due to the influence of resonances the actual motion can be considerably different from the one predicted by the averaging method. The two basic phenomena that are associated with the effect of a single resonance are capture into the resonance and scattering on the resonance.

Capture into a resonance can be described as follows. First the system evolves as it is predicted by the averaging method. At a certain time moment the system approaches a resonance. After that the system evolves in such a way that the resonant relation between frequencies is being kept approximately. As a result, after a time $1/\varepsilon$ the state of the system is completely different from the one predicted by the averaging method. Initial conditions for trajectories with a capture and trajectories without captures are mixed, if ε is small. Thus, it is reasonable to consider the capture as a random event and to calculate a probability of this event. Capture into a resonance was first discussed by P.Goldreich, S.Peale (1966), A.M.Moltchanov (1968) in connection with problems of celestial mechanics.

Scattering on resonances is observed for trajectories which pass through resonaces without being captured. Scattering is a deviation of such trajectory from that predicted by the averaging method. During a passage through a narrow neighborhood of the resonance, position of the system's phase point with respect to the trajectory of the averaged system undergoes a jump small. The amplitude of this jump is very sensitive to the variation of initial conditions, if ε is small. Thus, it is reasonable to consider this jump as random scattering on the resonance. Scattering on a resonance was first discussed by B.V.Chirikov (1959) and V.I.Arnold (1965).

In the talk we discuss calculation of probability of a capture into a resonance, description of motion in a state of a capture into resonance and calculation of a statistical distribution of the amplitude of a scattering on a resonance.

Again, the case when the perturbation is just a slow variation of parameters of the system in time or, more generally, the perturbed system is a slow-fast Hamiltonian system, is of special interest. For this case captures into resonances and scatterings on resonances lead to destruction of adiabatic invariance of "action" variables of unperturbed system. Similar phenomena take place in volume-preserving systems.