RESIDUAL STRESS FIELDS IN SOFT TISSUES

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Summary: We give a theory based on local stress free reference configurations, for identifying the residual strain in elastic materials. The residual strain is obtained as the solution of a minimization identification problem. As an example, we use this theory on a human aorta and the results agree well with other studies made on such an organ.

GENERAL THEORY

We begin with a description of a general theory for a residually stressed elastic material and end with a stress estimation for the aorta. The main goal is to find the residual strain that gives the residual stress. As a closing example, we use this framework on a human aorta. By optimization we can identify the residual strain and using that information we can calculate the stress and strain in the loaded configuration. In soft tissue mechanics it is important to calculate the residual stress that occurs in a body. This stress is defined as the stress that is left when all external loads are removed. The residual stress is important for the functionality of the tissue. For instance, it reduces stress gradients.

A loaded body configuration $\mathscr{B}$ and a stressed unloaded body configuration $\mathscr{B}_0$ are defined. The tangent map between $\mathscr{B}_0$ and $\mathscr{B}$ is described by a deformation gradient $\boldsymbol{F}$. To obtain a stress free reference configuration we assume that the unloaded configuration can be made stress free by releasing each material point and letting these points deform into a, in general, non compatible, configuration $\overline{\mathscr{B}}$. This stress free configuration can be described by a locally invertible two–point tensor $\boldsymbol{K}^{-1}$. The inverse $\boldsymbol{K}$ can be thought of as a tensor describing the initial strain for each material point and this strain gives a full description of the residual stress if the material is elastic.

The three configurations can be assigned different metric tensors. We define the metric tensors $\boldsymbol{\gamma}$, $\boldsymbol{G}$ and $\boldsymbol{g}$ on $\overline{\mathscr{B}}$, $\mathscr{B}_0$ and $\mathscr{B}$ respectively. We will also define an induced metric on $\mathscr{B}_0$ as

$$\boldsymbol{M} = \boldsymbol{K}^{-T}\boldsymbol{\gamma}\boldsymbol{K}^{-1}.$$

This metric is not necessarily Euclidean. The configuration $\mathscr{B}_0$ with metric $\boldsymbol{M}$ is a Riemannian manifold.

Many types of soft tissues, specially large arteries, can be approximated as incompressible materials. With this assumption the equilibrium equation without body forces becomes invariant with respect to the metric assigned to the configuration. This means that we can solve the equilibrium equation in $\mathscr{B}_0$ with the Euclidean metric $\boldsymbol{G}$ without loss of generality.

APPLICATION ON A HUMAN AORTA

As an example, we have used this theory on a human aorta to try to identify the initial strain tensor $\boldsymbol{K}$. The material behavior of the aorta is assumed to be orthotropic, hyper elastic and incompressible. The aorta is modeled as a thick walled cylinder so we introduce cylindrical coordinates (R,Θ,Z) in $\overline{\mathscr{B}}$ and (r,θ,z) in $\mathscr{B}$. In cylindrical coordinates the components of the metric $\boldsymbol{\gamma}$ and $\boldsymbol{g}$ are

$$\begin{aligned} &\gamma_{11} = 1 \quad \gamma_{22} = R^2 \quad \gamma_{33} = 1 \\ &g_{11} = 1 \quad g_{22} = r^2 \quad g_{33} = 1. \end{aligned}$$

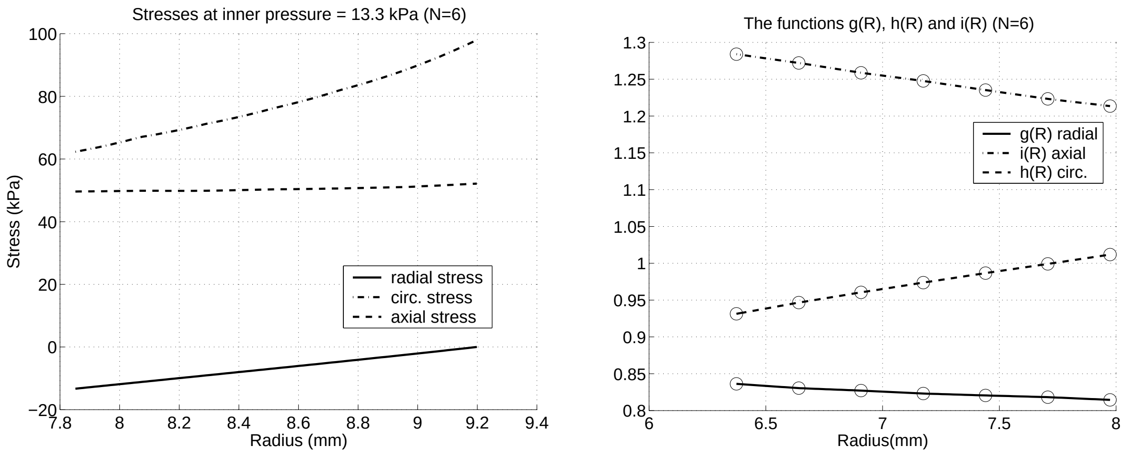
We also assume that the components of the strain tensor $\boldsymbol{K}$ form a diagonal matrix and that they are only dependent on the radial coordinate. Due to the incompressibility condition only two of these components are independent.

The equilibrium equation in $\mathscr{B}$, without external volume forces, and boundary conditions, can be written as

$$\begin{aligned} &\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0}, \\ &\boldsymbol{\sigma g n} = -P\boldsymbol{n} \quad r = r_0, \\ &\boldsymbol{\sigma g n} = \boldsymbol{0} \qquad\quad r = r_1, \end{aligned}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{n}$ a unit normal vector and r_0, r_1 is inner and outer radius, respectively, of the pressurized aorta wall. Solving this assuming that the stress is only dependent on the radial coordinate, we obtain a integral equation for the luminal pressure given by

$$P(r_i) = \int_{r_i}^{r_1} \left(r\sigma^{11} - \frac{1}{r}\sigma^{22} \right) \,\mathrm{d}r$$



(a) (b)

Figure 1: (a) Cauchy stresses in $\mathscr{B}$, (b) components of the initial strain tensor $\boldsymbol{K} \sim \operatorname{diag}(g(R), h(R), i(R))$ where R is defined in $\overline{\mathscr{B}}$. $N = 6$ is the number of linear functions that comprises $g(R),\ h(R)$ and $i(R)$. The circles in (b) denotes the interpolation points of $\boldsymbol{K}$.

where σ^{11} and σ^{22} are the components of the Cauchy stress tensor. By introducing a constitutive relation for the Cauchy stress that is dependent on the total deformation, we can calculate the inner pressure of the aorta wall. In this example we have chosen the constitutive relation suggested by Holzapfel et. al. [1]. The model parameters included in the constitutive equation can be determined by solving a minimization problem. The components of the initial strain tensor $\boldsymbol{K}$ are also unknown and must be approximated. In this case we have chosen to use continuous C^0 functions with equidistant interpolation nodes, fig. 1(b). This gives a optimization problem were the number of variables is equal to the sum of the number of material parameters and the number of interpolation nodes.

OPTIMIZATION AND RESULTS

Model parameters and residual strains can be obtained as a minimization problem were we try to fit the calculated pressure to measured data points. In this case we treated the minimization as a least square problem, which is formulated as

$$\begin{aligned} &\min_{\boldsymbol{Z},\boldsymbol{K}} ||P(r_i) - \tilde{P}_i||^2 \\ &\text{s.t.} \quad \underline{\boldsymbol{z}} \leq \boldsymbol{Z} \leq \overline{\boldsymbol{z}} \\ &\qquad\ \underline{\boldsymbol{k}} \leq \boldsymbol{K} \leq \overline{\boldsymbol{k}} \end{aligned}$$

where $\tilde{P}_i$ is the measured pressure at inner radii r_i, $\boldsymbol{Z}$ are the model parameters from the constitutive relation and $\boldsymbol{K}$ are the interpolation values for the initial strain. The over- and underline represent the lower and upper bound on the optimization parameters, respectively. The data was taken from an in–vivo study on the human aorta and consists of pressure and inner radii of the aorta wall. The pressure $\tilde{P}$ was measured invasively using a catheter and the inner radius by ultrasound. The magnitude of the calculated stresses are in agreement with an in–vivo study on the human aorta made by Schulze–Bauer and Holzapfel [2], fig. 1(a). The functional form of the components of the initial strain tensor $\boldsymbol{K}$ are close to linear, see fig. 1 (b). A calculation of the Green–Lagrange strains at 13.3 kPa shows that the circumferential strain is close to constant through the wall. This is in agreement with the generally accepted uniform strain hypothesis.

[1] Holzapfel G.A., Gasser T.C., Ogden R.W., A new constitutive framework for arterial wall mechanics and a comparative study of material models, *J. Elasticity*, 61, 1–48, 2000

[2] Schulze-Bauer C.A.J., Holzapfel G.A., Determination of constitutive equations for human arteries from clinical data, *J. Bio. Mech.*, 36, 165–169, 2003