TRANSIENT EIGENSTRAINS WITHOUT INCREMENTAL DISPLACEMENTS IN A HYPERELASTIC BODY

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Summary We consider a hyperelastic body in a static intermediate configuration, loaded by an additional distribution of transient eigenstrains, the latter being e.g. due to the electric field in a piezoelastic body, or due to the temperature in a thermoelastic body. We seek for distributions of eigenstrain, such that that the incremental displacements vanish throughout the body. In the present paper, within the framework of the theory of small dynamic deformations superimposed upon a static strain, we derive solutions for eigenstrains that are applied at a sub-region of the body only.

1. INTRODUCTION

Consider a structure that is fixed at some part of its boundary, and that is initially at rest in a pre-deformed intermediate state with a possibly large strain. The constitutive behavior of this body is taken as hyperelastic, and the static intermediate state is assumed to be infinitesimally superstable. The body is loaded by an additional distribution of transient eigenstrains, the actuating effects of which are described by actuation stresses. Eigenstrains e.g. may be due to the electric field in a piezoelastic body, or due to the temperature in a thermoelastic body. We seek for additional actuation stresses, such that that the incremental displacements vanish throughout the body for all times. The body then will remain in the static intermediate configuration, despite its transient actuation by eigenstrains. A static solution for such "nilpotent" distributions was addressed by Irschik and Ziegler [1], and the dynamic extension was given in Irschik and Pichler [2], see Irschik et. al. [3] for beam vibrations. In these solutions, the actuation was required to be distributed all over the body under consideration, and the linear theory of elasticity was employed. In the present paper, we derive solutions for actuation stresses that are applied at a sub-region of the body only, and we work within the framework of the theory of small dynamic deformations superimposed upon a possibly large static strain.

2. A THEOREM ON EIGENSTRAINS WITHOUT DISPLACEMENTS

Consider an elastic body being at rest in some deformed state of equilibrium. This state may represent a large static deformation from an undistorted configuration of the body and is denoted as the intermediate configuration. Due to an additional time-dependent eigenstrain, small incremental displacements u are superimposed upon the intermediate configuration, accompanied by small incremental deformations. In order to describe this small motion, we use the intermediate state as the reference configuration. The incremental first Piola-Kirchhoff stress tensor S then is given as:

$$S = A[H] + S^a \tag{1}$$

with $S = T_R - T$, where T_R is the first Piola-Kirchhoff stress in the actual configuration and T is the static Cauchy stress in the intermediate configuration. The gradient of u with respect to the place in the intermediate configuration is denoted as H. The second order tensor S^a in Eq. (1) represents the incremental stress actuation tensor due to the additional eigenstrain superimposed upon the intermediate state. E.g., in case this eigenstrain is due to an incremental temperature rise Θ, the actuation stress can be taken as a linear mapping of Θ, since we deal with small incremental displacements, $S^a = A[\Theta M]$, the quantity ΘM representing the tensorial eigenstrain. When the additional eigenstrain is formed by an electric field in a piezoelastic body, S^a can be also written as a linear mapping of the electric field vector. The incremental stress S has to satisfy the local form of the equation of balance of linear momentum,

$$DivS = \rho \ddot{u}, \tag{2}$$

where a superposed dot denotes the time-derivative, and ρ is the mass density. Div stands for the divergence operator with respect to the place in the intermediate configuration. Homogeneous kinematic boundary conditions are assigned at the part ∂B_1 of the boundary, while at the remaining part ∂B_2 the additional surface tractions are taken to vanish. Since we assume the additional motion to start from the intermediate configuration, which is at rest, we deal with homogeneous initial conditions. In the present contribution, we derive the following theorem:

Assume that the intermediate configuration is infinitesimally superstable and that the additional actuation stress S^a acts in some sub-region $\hat{B}$ of the body B, vanishing outside of $\hat{B}$. Consider the class of a non-vanishing divergence-free actuation stress in $\hat{B}$ satisfying homogeneous boundary conditions at $\partial\hat{B}_2$, a so-called statically admissible stress,

$$DivS^a = 0, \qquad \partial\hat{B}_2 : \; S^a \hat{n} = 0. \tag{3}$$

This class produces a displacement that vanishes throughout B, $u = 0 \;\; \forall \;\; t$. Hence, when the actuating stress S^a satisfies Eqs.(3), then the incremental displacements u vanish throughout the body B for all times $t \geq 0$. The body thus will remain in the static intermediate configuration, despite the transient action of S^a.

3. NUMERICAL EXAMPLE

The above result for eigenstrains without displacements should be of technological interest in various fields, due to the widespread applicability of the theory of eigenstrains. The methodology requires that the actuation stress be tailored, which becomes practically visible in view of modern functional and smart materials. In the following, we present a computational example using the Finite Element Code ABAQUS, which allows taking into account a non-homogeneously distributed anisotropic transient actuation stress of thermal origin. Consider the quadratic domain B shown in Fig.1 in a state of plane strain. The region is subdivided into a sufficiently high number of Finite Elements of the type CPE4R. Two adjacent edges, EAC, of B are fixed, forming the part ∂B_1 of the boundary. The remaining edges CDE form the stress-free part ∂B_2 of the boundary. The rectangle $EAFG$ forms the subregion $\hat{B}$. We assume the region B to represent an undistorted, stress-free configuration, with the isotropic tensor of elastic parameters A (Young's modulus $Y = 211.10^6$ in proper units of stress, and Poisson's ratio □ = 0.3).

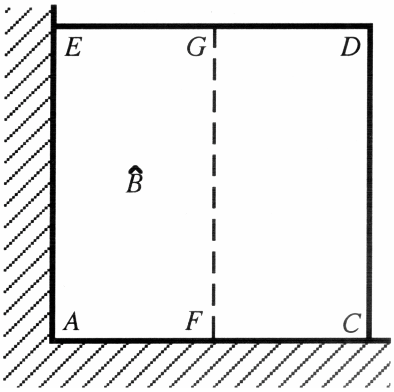

Figure 1: Original problem

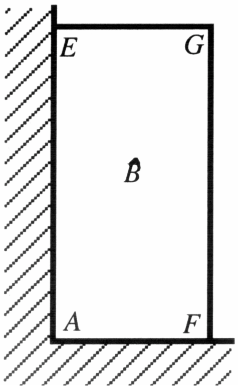

Figure 2: Auxiliary problem

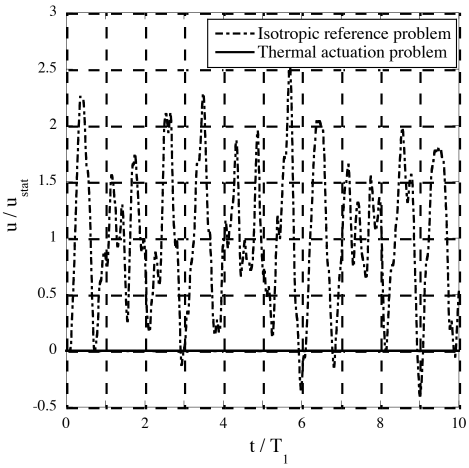

Figure 3: Step response

In order to obtain a non-vanishing reference displacement, we consider the case of a transient isotropic actuation stress of thermal origin, being distributed constantly within the subdomain $\hat{B}$ in Fig. 1, $S_i^a = \hat{S}^a H(t)$, $\hat{S}^a =$ □□□$_o A[I]$, and vanishing outside of $\hat{B}$, in the domain $GFCD$. $H(t)$ denotes the Heaviside step function, the unit tensor is denoted as I, and the thermal eigenstrain is chosen as □□$_o = 2.10$□9. The corresponding step-response displacements are denoted as isotropic reference problem in Fig.3. We then consider the problem of determining a statically admissible stress, see Eqs.(3), in order to obtain a vanishing displacement. For that sake, we treat the auxiliary problem of Fig.2, where the fixed part $\partial\hat{B}_1$ of the corresponding body with domain $\hat{B}$ is formed by the adjacent edges EAF, and the stress-free part $\partial\hat{B}_2$ is formed by FGE. The static actuation stress $\hat{S}^a$ is applied to the auxiliary body shown in Fig. 2, and the corresponding static stress distribution $\hat{S}$ is computed. Following the above theorem, the transient actuation stress $S^a = \hat{S}\ H(t)$, which is anisotropic and non-homogeneously distributed, then is applied to the subdomain $\hat{B}$ of B in the original problem, since such an S^a satisfies Eqs.(3). The step response of the domain B is again calculated by means of the dynamic Finite Element code. As a typical result, the horizontal displacement of corner D is presented in Fig. 3 (denoted as thermal actuation solution) and compared to the corresponding step response of the isotropic reference problem. It is seen that the dynamic displacement due to S^a (the thermal actuation solution) is not only much smaller than the isotropic reference solution, but that it indeed does vanish, as predicted by our above theorem.

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