# EXPLOITATION OF INCREMENTAL ENERGY MINIMIZATION PRINCIPLES IN COMPUTATIONAL MULTISCALE ANALYSES OF INELASTIC SOLIDS

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*Summary* The lecture provides an overview about recent developments in the formulation and numerical implementation of incremental minimization principles for inelastic solids and their exploitation with regard to multiscale analyses of deformation microstructures. The incremental energy minimization is applied to conceptual model problems which treat multiscale microstructure evolutions in stable / instable and a priori heterogeneous / homogeneous elastic–plastic solids.

## Introduction

The lecture overviews recent developments in the formulation and numerical implementation of incremental minimization principles for inelastic solids and their exploitation with regard to multiscale analyses of deformation microstructures. Key aspect is the outline of *minimization principles for standard dissipative materials* that govern the microstructure development of a priori heterogeneous materials as well as deformation phase decompositions in initially homogeneous materials. These principles allow to recast incremental initial boundary–value problems for inelastic solids into minimization problems. The minimization structure provides a *canonical approach to inelastic solid mechanics under quasistatic conditions*, with important consequences to almost all subsequent aspects of the modelling and the numerical implementation: • The existence of solutions of the incremental IBVPs depends on weak convexity properties of a stress potential. • Stress update algorithms in plasticity, viscoplasticity and damage mechanics appear in a natural format in the form of energy minimizers. • Micro–to–macro transitions for the modelling of the overall response of a priori given heterogeneous microstructures can be defined in terms of a principle of minimum averaged incremental energy. • The material stability of incremental plastic deformations can be defined in terms of the quasi–convexity of incremental stress potentials in analogy to elasticity, which provides a more general criterion than the classical Hadamard condition. • Deformation micro–structure developments in non–stable solids can be interpreted as phase decompositions and be resolved by energy minimization principles of quasi– and rank–one convexifications of the incremental stress potential.

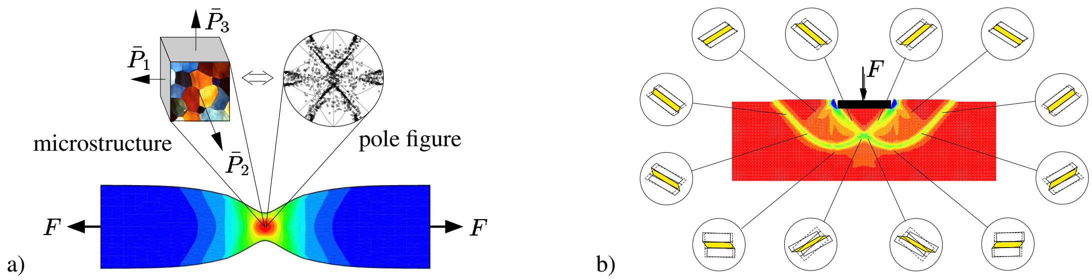


**Figure 1.** Examples of multiscale microstructural analyses in finite plasticity. a) Homogenization analysis predicts texture development in heterogeneous polycrystals. b) Relaxation analysis of instable solids governs the evolution of postcritcal deformation phases.

## Incremental Constitutive Minimization Problem (C)

The point of departure is a general internal variable formulation for the constitutive response of standard dissipative materials in terms of an energy storage and a dissipation function. Nonlinear standard materials cover a broad spectrum of constitutive models in finite elasticity, viscoelasticity, plasticity or damage mechanics. Consistent with this type of finite inelasticity we outline a distinct *incremental variational formulation of the local constitutive response* as proposed in [2] and [5], where a quasi–hyperelastic stress potential is obtained from a local minimization problem with respect to the internal variables. It is shown that this local minimization problem determines the internal state of the material for finite increments of time. The approach is conceptually in line with works on incremental variational formulations and variational updates in plasticity by [7] and [1].

## Incremental Minimization Problem of Relaxation (R)

The existence of this variational formulation allows the definition of the *material stability of an inelastic solid based on weak convexity conditions* of the incremental stress potential in analogy to classical treatments of finite elasticity. Furthermore, material instability phenomena may be interpreted as deformation microstructure developments associated with non–convex incremental stress potentials in analogy to elastic phase decomposition problems. These micro–structures can be resolved by the relaxation of non–convex energy functionals based on a convexification of the stress potential. The relaxed problem provides a well–posed formulation for a mesh–objective analysis as close as possible to the non–convex original problem. Following the recent work [4], we give a conceptual outline of a relaxation analysis for non–convex standard dissipative solids and specify a procedure to a one–level rank–one convexification procedure.

- **Constitutive Model (M).** $\boldsymbol{F} \in GL_{+(3)}$ at $\boldsymbol{X} \in \mathcal{B}$ is the local deformation of the material and $\boldsymbol{\mathcal{I}} \in \mathcal{G}$ a generalized vector of internal variables. A normal–dissipative set of local material equations has the structure

  *stresses* $\quad \boldsymbol{P} = \partial_{\boldsymbol{F}} \psi(\boldsymbol{F}, \boldsymbol{\mathcal{I}}; \boldsymbol{X})$

  *evolution equation* $\quad \partial_{\boldsymbol{\mathcal{I}}} \psi(\boldsymbol{F}, \boldsymbol{\mathcal{I}}; \boldsymbol{X}) + \partial_{\dot{\boldsymbol{\mathcal{I}}}} \phi(\dot{\boldsymbol{\mathcal{I}}}, \boldsymbol{\mathcal{I}}; \boldsymbol{X}) = 0 \quad , \boldsymbol{\mathcal{I}}(0) = \boldsymbol{\mathcal{I}}_0$

  defined in terms of an energy storage function $\psi$ and a dissipation function $\phi$.

- **Incremental Variational Formulation of Constitutive Model (C).** In a finite time increment $[t_n, t_{n+1}]$, the *minimization problem of the constitutive response*

  *stresses* $\quad \boldsymbol{P}_{n+1} = \partial_{\boldsymbol{F}} W(\boldsymbol{F}_{n+1}; \boldsymbol{X})$

  *stress potential* $\quad W(\boldsymbol{F}_{n+1}; \boldsymbol{X}) = \inf_{\boldsymbol{\mathcal{I}}} \int_{t_n}^{t_{n+1}} [\, \dot{\psi} + \phi \,] \, dt \quad , \boldsymbol{\mathcal{I}}(t_n) = \boldsymbol{\mathcal{I}}_n$

  determines the current internal state $\boldsymbol{\mathcal{I}}_{n+1} \in \mathcal{G}$ and provides a potential for the stresses at time $t_{n+1}$. Approximation by variational update algorithm.

- **Stability of Incremental Constitutive Response (S).** In $[t_n, t_{n+1}]$ the *material is locally stable if the incremental stress potential $W$ is quasi–convex*

  *stable response* $\quad \frac{1}{|D|} \int_D W(\boldsymbol{F}_{n+1} + \nabla \boldsymbol{w}(\boldsymbol{y})) \, dV \geq W(\boldsymbol{F}_{n+1})$

  for all possible fluctuations $\boldsymbol{w}(\boldsymbol{y})$ on the domain $D$.

- **Microstructure Development in Homogeneous Materials (R).** For an instable non–convex response, the *incremental minimization problem of convexification*

  *macro–stresses* $\quad \boldsymbol{P}_{Qn+1} = \partial_{\boldsymbol{F}} W_Q(\boldsymbol{F}_{n+1})$

  *relaxation* $\quad W_Q(\boldsymbol{F}_{n+1}) = \inf_{\boldsymbol{w}} \frac{1}{|\mathcal{V}|} \int_{\mathcal{B}} W(\boldsymbol{F}_{n+1} + \nabla \boldsymbol{w}(\boldsymbol{y})) dV$

  provides a relaxed convex hull $W_Q$ of $W$ and determines the current microstructure fluctuation field $\boldsymbol{w}(\boldsymbol{y})$. Possible approximation by first–order rank–one relaxation.

- **Microstructure Development in Heterogeneous Materials (H).** $\bar{\boldsymbol{F}}_{n+1} \in GL_+(3)$ is the macro–deformation of a heterogeneous microstructure $\mathcal{B}$ at time $t_{n+1}$. The *incremental minimization problem of homogenization*

  *macro–stresses* $\quad \bar{\boldsymbol{P}}_{n+1} = \partial_{\bar{\boldsymbol{F}}} \bar{W}(\bar{\boldsymbol{F}}_{n+1})$

  *homogenization* $\quad \bar{W}(\bar{\boldsymbol{F}}_{n+1}) = \inf_{\boldsymbol{w}} \frac{1}{|\mathcal{V}|} \int_{\mathcal{B}} W(\bar{\boldsymbol{F}}_{n+1} + \nabla \boldsymbol{w}(\boldsymbol{X}); \boldsymbol{X}) dV$

  determines the homogenized potential $\bar{W}$ and the fluctuation field $\boldsymbol{w}(\boldsymbol{X})$ on the heterogeneous microstructure. Possible approximation by a finite element discretization.

**Figure 2.** Incremental energy minimization principles of dissipative solids for multiscale homogenization and relaxation analyses.

## Incremental Minimization Problem of Homogenization (H)

The existence of the quasi–hyperelastic incremental stress potentials for standard dissipative materials also allows the extension of homogenization approaches of finite elasticity to the incremental setting of inelasticity. Following the works [5] and [3], we outline an *incremental variational formulation of the homogenization problem* for standard dissipative materials at finite strains, where a quasi–hyperelastic macro–stress potential is obtained from a global minimization problem with respect to the fine–scale displacement fluctuation field. It is shown that this global minimization problem determines the state of the heterogeneous micro–structure for finite increments of time.

In the lecture successivly treats theoretical and algorithmic formulations of the above mentioned minimization problems (C), (R) and (H), where Figure 2 provides a guide. The performance of the formulations is demonstrated for conceptual model problems of multiscale microstructure developments in stable and instable materials as indicated in Figure 1.

## References


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