AN UPDATED ARBITRARY-LAGRANGIAN-EULERIAN DESCRIPTION IN CONTINUUM MECHANICS AND ITS APPLICATION TO NONLINEAR FLUID-STRUCTURE INTERACTION DYNAMICS

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Introduction

The applications of finite element methods (FEM) in structural analysis [1-2] and computational fluid dynamics (CFD) [3-4] have achieved increased sophistication and success in recent years. However, when these methods are used to solve problems in nonlinear fluid-solid interaction dynamics (NFSID), problems arise. For example, in a Lagrangian description of solids, variables describing structural motion are functions of the material coordinates fixed to each particle of the solid whereas in an Eulerian description of fluids, variables describing the fluid flow are functions of the coordinates fixed in space. When the structure moves, the material coordinate points also move from their original positions to new positions in space but, in the surrounding fluid, the Eulerian coordinates remain unchanged. On fluid-structure interaction interfaces, this difference of description produces a separation of the mesh points defining the solid from those of the fluid, which coincided before the system was disturbed. To retain the validity of compatibility conditions on the coupling interfaces in (NFSID), an Arbitrary-Lagrangian-Eulerian (ALE) system [5] has been adopted to describe the fluid motions [6-10]. The mathematical models reported include the FEM model for both fluids and solids and combination models adopting a FEM in solids and a finite volume (FV) or finite difference (FD) approaches in fluids. To solve NFSID problems, FD/FV techniques of CFD and FEM techniques for solids are used and the developed numerical method is referred as a *partitioned* or *iterative* solution procedure.

In these ALE descriptions, the reference coordinates remain unchanged during fluid motion and therefore form a curvilinear system of which the base vectors of the system are functions of time and space. The consequence of this is that all mathematical operations and formulations involving derivatives, volume and area transformations, etc are very complex functions and not readily adaptable to a numerical analysis [11]. In nonlinear FEA analysis of solids, an Updated Lagrangian (UL) system has been developed and successfully used to deal with large motions of the solid [1-2]. For an UL system, the current position coordinates (CPC) of each material point are chosen as its Lagrangian coordinates to describe the motions. With the marching of time, the material point moves to a new position and its Lagrangian coordinates are updated to its new CPC. Following this idea, this paper proposes and develops an updated ALE (UALE) system in which the CPC of the reference point in CESS are chosen as the reference coordinates to describe the motions of the fluids. As time advances, the reference point moves to its new position and its reference coordinates are updated to its new CPC. To fully understand the proposed concept, geometrical and physical explanations for this UALE system are described. In development of the theoretical model, derivations and formulations are included of mathematical operation rules for time and spatial derivatives, differential and integral conservation laws in continuum mechanics and variational formulations in the UALE system. A variational approximate solution is presented and a mixed FEM-FV method to NFSID problems developed to demonstrate the applications of this UALE system.

UALE system

In a Cartesian Eulerian spatial system (CESS) $o-x_1x_2x_3$ with base vectors $\mathbf{a}_i$, a mesh point *p,* located at a position $\mathbf{r}^0 = x_i^0\mathbf{a}_i$ at time $t=0$, moves to a new point p' with position vector $\mathbf{r} = x_i\mathbf{a}_i$ at time $t=t_1$. In a total ALE system, the original position coordinates of the mesh point *p* are chosen as the reference coordinates, i.e. $\xi^i = x_i^0$. During the process of motion, the reference coordinates ξ^i remain unchanged, although a mesh line undergoes deformation. The base vectors at the mesh point ξ^i of the mesh system at time $t=0$ and $t=t_1$ are $\mathbf{g}_i^0 = \partial\mathbf{r}^0/\partial\xi^i = \mathbf{a}_i$ and $\mathbf{g}_i^t = \partial\mathbf{r}/\partial\xi^i = \mathbf{a}_j\partial x_j/\partial\xi^i$, respectively. However, an UALE system chooses the current position coordinates x_i to denote a mesh point. At time $t=0$, the mesh coordinates are $z_i = x_i^0$ and the base vectors at the point *p* are $\mathbf{g}_i^0 = \mathrm{a}_\mathrm{i}$. At time $t=t_1$, the mesh point *p* moves to a new position P', which now has an *updated* mesh coordinate

$z_i = x_i$. The base vectors at p' of the updated system at time $t = t_1$ are given by $\mathbf{g}_i^t = \partial \mathrm{r} / \partial z_i = \partial x_j / \partial x_i \mathrm{a}_j = \mathrm{a}_i$. Therefore, the UALE system has the same base vectors as the CESS during all motions. The *updated velocity of the reference coordinates* is shown to be the moving velocity of the mesh in the ALE system. The UL system can be considered as a special case of an UALE system in which the reference point is replaced by a material point.

Conservation laws, mathematical operations and variational principles for NFSID

The differential and integral forms of the generalized conservation laws in an UALE system are derived and the mathematical operation rules applicable to derivatives, integral and variations are presented. The transformation relations between ALE and UALE systems are provided. A *Eulerian virtual displacement* of a material point in a spatial description and its *relative virtual displacement* (i.e., *convective* and *local virtual displacements*) of the material point in the ALE reference description are defined. For a field function, the material, Eulerian local and ALE reference local variations as well as their transformation relations are formulated. For a functional $H[\phi]$ defined over the region $\Omega(x_i, t)$, the material, Eulerian local and ALE reference local variations of this functional are given. In the UALE system, special variational relations are derived. From these relations, variational principles for NFSID in an UALE system are obtained and compared with the forms appertaining to spatial coordinate systems [12]. An example of a variational approximate solution to a NFSID problem is presented. These variational principles provide a basis to construct numerical schemes of study in the UALE system.

A mixed FEM-FV approach

Both UALE and UL systems choose the same CPC in CESS as the updated reference or updated material coordinates and have the same base vectors and updated velocity. This provides benefits when dealing with mesh matching difficulties in the numerical analysis of NFSID problems and allows development of a mixed FEM-FV. That is, an UL incremental formulation [1-2] for solids and a FV method for fluids are used. The integral forms of the conservation laws of the fluid developed in UALE system are easily related to the computational grid, where the volume integrals are taken over a current finite volume. A coupling iteration is used to solve the coupling equations using FEA and FV software. On the assumption that the solution of the fluid, solid and all variables in the fluid-solid interaction problem at time *t* have been derived in the previous cycle, the configuration of the system at time $t + \Delta t$ needs to be calculated. The developed coupling iteration solution process starts by solving the solid equation using prescribed load intensities at time $t + \Delta t$, the fluid stress on the wetted interface and an integration over the volume and area last calculated to predict load vectors and to derive an approximate displacement. The FV calculation for the fluid is now completed by evaluating an approximation of the fluid configuration and field variables, which are in turn used in the solid analysis to check convergence and to determine if the iteration process may proceed.

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