VIEWS ON MATERIAL FORCES IN MULTIPLICATIVE ELASTOPLASTICITY

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Summary The main goal of this contribution is the examination of a general framework for finite hyper–elastoplasticity that reflects the nature of material forces. In particular, we thereby address representations of Eshelbian stress tensors and Eshelbian volume forces with respect to different configurations, namely the spatial, the material and – what we call – the intermediate setting which allows alternative interpretation as being referred to a local rearrangement. Deriving these relations, one naturally incorporates connections which are determined by either the irreversible or the reversible portion of the deformation gradient. The physical interpretation of these contributions consists in the fact that their skew part can be related to the, say, dislocation density. With these Eshelbian stress tensors and volume forces at hand, we finally come up with different representations of balances of linear momentum which are carried out with respect to the spatial or material setting, referring either to the spatial or to the material motion problem. The developed framework serves as the fundamental outset for the application of the material force method.

ESSENTIAL KINEMATICS

Let the deformation gradient of the (sufficiently smooth) spatial motion problem, $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$ in $\mathcal{B}_t$, be decomposed via $\mathrm{D}_{\boldsymbol{X}}\boldsymbol{\varphi} = \boldsymbol{F} \doteq \boldsymbol{F}_\mathrm{e} \cdot \boldsymbol{F}_\mathrm{p}$. The corresponding tangent map of the material motion problem, $\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x}, t)$ in $\mathcal{B}_0$, consequently reads $\mathrm{d}_{\boldsymbol{x}}\boldsymbol{\Phi} = \boldsymbol{f} \doteq \boldsymbol{f}_\mathrm{p} \cdot \boldsymbol{f}_\mathrm{e}$ whereby $\boldsymbol{f} \doteq \boldsymbol{F}^{-1}$, $\boldsymbol{f}_\mathrm{e} \doteq \boldsymbol{F}_\mathrm{e}^{-1}$ and $\boldsymbol{f}_\mathrm{p} \doteq \boldsymbol{F}_\mathrm{p}^{-1}$. For completeness, let appropriate velocity fields be denoted by $\boldsymbol{v} = \mathrm{D}_\mathrm{t}\boldsymbol{\varphi}$ with $\boldsymbol{l} = \mathrm{d}_{\boldsymbol{x}}\boldsymbol{v} = \mathrm{D}_\mathrm{t}\boldsymbol{F} \cdot \boldsymbol{f}$ and $\boldsymbol{V} = \mathrm{d}_\mathrm{t}\boldsymbol{\Phi}$ with $\boldsymbol{L} = \mathrm{D}_{\boldsymbol{X}}\boldsymbol{V} = \mathrm{d}_t\boldsymbol{f} \cdot \boldsymbol{F}$ as well as $\boldsymbol{v} = -\boldsymbol{F} \cdot \boldsymbol{V}$.

BALANCES OF LINEAR MOMENTUM

The classical format of balance of linear momentum is usually outlined in terms of, e.g., the spatial motion first Piola–Kirchhoff stress tensor, $\boldsymbol{\Pi}^\mathrm{t}$. When referring to both, the spatial as well as to the material motion problem (here for the static case) we end up with different representations which are related via Piola transformations, to be specific

$$\begin{array}{llllll} \nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Pi}^\mathrm{t} + \boldsymbol{b}_0 = \boldsymbol{0}\,, & \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma}^\mathrm{t} + \boldsymbol{b}_t = \boldsymbol{0} & \text{with} & \boldsymbol{\sigma}^\mathrm{t} = \det(\boldsymbol{f})\,\boldsymbol{\Pi}^\mathrm{t} \cdot \boldsymbol{F}^\mathrm{t} & \text{and} & \boldsymbol{b}_t = \det(\boldsymbol{f})\,\boldsymbol{b}_0 \quad (\text{in } T\mathcal{B}_t) \\ \nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Sigma}^\mathrm{t} + \boldsymbol{B}_0 = \boldsymbol{0}\,, & \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\pi}^\mathrm{t} + \boldsymbol{B}_t = \boldsymbol{0} & \text{with} & \boldsymbol{\pi}^\mathrm{t} = \det(\boldsymbol{f})\,\boldsymbol{\Sigma}^\mathrm{t} \cdot \boldsymbol{F}^\mathrm{t} & \text{and} & \boldsymbol{B}_t = \det(\boldsymbol{f})\,\boldsymbol{B}_0 \quad (\text{in } T\mathcal{B}_0) \end{array} \tag{1}$$

DISSIPATION INEQUALITY

Let the free Helmholtz energy take the format

$$\psi_0 = \psi_0(\boldsymbol{F}, \boldsymbol{F}_\mathrm{p}; \boldsymbol{X}) = \det(\boldsymbol{F}_\mathrm{p})\,\psi_\mathrm{p}(\boldsymbol{F} \cdot \boldsymbol{f}_\mathrm{p}; \boldsymbol{X}) \tag{2}$$

such that the (isothermal) Dissipation inequality of the spatial motion problem, $\mathsf{D}_0^\mathrm{loc} = \boldsymbol{\Pi}^\mathrm{t} : \mathrm{D}_t\boldsymbol{F} - \mathrm{D}_t\psi_0 \geq 0$, reads

$$\mathsf{D}_0^\mathrm{loc} = [\,\boldsymbol{\Pi}^\mathrm{t} - \mathrm{D}_{\boldsymbol{F}}\psi_0\,] : \mathrm{D}_t\boldsymbol{F} - \mathrm{D}_{\boldsymbol{F}_\mathrm{p}}\psi_0 : \mathrm{D}_t\boldsymbol{F}_\mathrm{p} \doteq -\,\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t} : \mathrm{D}_t\boldsymbol{F}_\mathrm{p} = -\,\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t} : [\,\mathrm{d}_t\boldsymbol{F}_\mathrm{p} - \nabla_{\boldsymbol{X}}\boldsymbol{F}_\mathrm{p} \cdot \boldsymbol{V}\,] \geq 0 \tag{3}$$

When placing emphasis on the definition of the force driving $\mathrm{D}_t\boldsymbol{F}_\mathrm{p}$ one observes that $\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t}$ takes an almost Eshelbian format, namely

$$\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t} = \mathrm{D}_{\boldsymbol{F}_\mathrm{p}}\psi_0 = \psi_\mathrm{p}\,\mathrm{D}_{\boldsymbol{F}_\mathrm{p}}\det(\boldsymbol{F}_\mathrm{p}) + \det(\boldsymbol{F}_\mathrm{p})\,\mathrm{D}_{\boldsymbol{F}_\mathrm{e}}\psi_\mathrm{p} : \mathrm{D}_{\boldsymbol{F}_\mathrm{p}}\boldsymbol{F}_\mathrm{e} = \psi_0\,\boldsymbol{f}_\mathrm{p}^\mathrm{t} - \det(\boldsymbol{F}_\mathrm{p})\,\boldsymbol{F}_\mathrm{e}^\mathrm{t} \cdot \mathrm{D}_{\boldsymbol{F}_\mathrm{e}}\psi_\mathrm{p} \cdot \boldsymbol{f}_\mathrm{p}^\mathrm{t} \tag{4}$$

with $\mathrm{D}_{\boldsymbol{F}_\mathrm{e}}\psi_\mathrm{p} = \mathrm{D}_{\boldsymbol{F}}\psi_\mathrm{p} : \mathrm{D}_{\boldsymbol{F}_\mathrm{e}}\boldsymbol{F} = \mathrm{D}_{\boldsymbol{F}}\psi_\mathrm{p} \cdot \boldsymbol{F}_\mathrm{p}^\mathrm{t}$ and $\det(\boldsymbol{F}_\mathrm{p})\,\mathrm{D}_{\boldsymbol{F}}\psi_\mathrm{p} = \mathrm{D}_{\boldsymbol{F}}\psi_0 = \boldsymbol{\Pi}^\mathrm{t}$ which results in

$$\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t} = \psi_0\,\boldsymbol{f}_\mathrm{p}^\mathrm{t} - \det(\boldsymbol{F}_\mathrm{p})\,\boldsymbol{F}_\mathrm{e}^\mathrm{t} \cdot \mathrm{D}_{\boldsymbol{F}}\psi_\mathrm{p} = \psi_0\,\boldsymbol{f}_\mathrm{p}^\mathrm{t} - \boldsymbol{F}_\mathrm{e}^\mathrm{t} \cdot \boldsymbol{\Pi}^\mathrm{t} = [\,\psi_0\,\boldsymbol{I}_\mathrm{p} - \boldsymbol{F}_\mathrm{e}^\mathrm{t} \cdot \boldsymbol{\Pi}^\mathrm{t} \cdot \boldsymbol{F}_\mathrm{p}^\mathrm{t}\,] \cdot \boldsymbol{f}_\mathrm{p}^\mathrm{t} = \boldsymbol{\Sigma}_\mathrm{p}^\mathrm{t} \cdot \boldsymbol{f}_\mathrm{p}^\mathrm{t} \tag{5}$$

whereby $\boldsymbol{F}_\mathrm{e}^\mathrm{t} \cdot \boldsymbol{\Pi}^\mathrm{t} \cdot \boldsymbol{F}_\mathrm{p}^\mathrm{t} = \boldsymbol{M}_\mathrm{p}^\mathrm{t}$ characterizes a Mandel stress tensor of the spatial motion problem. With this relation in hand, it is now straightforward to show that the Eshelbian stress field $\boldsymbol{\Sigma}_\mathrm{p}^\mathrm{t}$ is the driving force of the plastic velocity gradient $\boldsymbol{L}_\mathrm{p} = \mathrm{D}_t\boldsymbol{F}_\mathrm{p} \cdot \boldsymbol{f}_\mathrm{p} = -\,\boldsymbol{F}_\mathrm{p} \cdot \mathrm{D}_t\boldsymbol{f}_\mathrm{p}$, i.e.

$$\mathsf{D}_0^\mathrm{loc} = -\,[\,\boldsymbol{\Pi}_\mathrm{p}^\mathrm{t} \cdot \boldsymbol{F}_\mathrm{p}^\mathrm{t}\,] : [\,\mathrm{D}_t\boldsymbol{F}_\mathrm{p} \cdot \boldsymbol{f}_\mathrm{p}\,] = -\,\boldsymbol{\Sigma}_\mathrm{p}^\mathrm{t} : \boldsymbol{L}_\mathrm{p} \geq 0 \tag{6}$$

In view of the material motion problem, we first define the free Helmholtz energy as

$$\psi_t = \psi_t(\boldsymbol{F}, \boldsymbol{F}_\mathrm{p}; \boldsymbol{X}) = \det(\boldsymbol{F}_\mathrm{p} \cdot \boldsymbol{f})\,\psi_\mathrm{p}(\boldsymbol{F} \cdot \boldsymbol{f}_\mathrm{p}; \boldsymbol{X}) \tag{7}$$

and second exploit the (isothermal) dissipation inequality of the material motion problem, $\mathsf{D}_t^{\mathrm{loc}} = \det(\boldsymbol{f})\, \mathsf{D}_0^{\mathrm{loc}}$, in detail

$$\mathsf{D}_t^{\mathrm{loc}} = \boldsymbol{\pi}^{\mathrm{t}} : \mathrm{d}_t \boldsymbol{f} - \boldsymbol{B}_t^{\mathrm{int}} \cdot \boldsymbol{V} - \mathrm{d}_t \psi_t = [\, \boldsymbol{\pi}^{\mathrm{t}} - \mathrm{d}_{\boldsymbol{f}} \psi_t \,] : \mathrm{d}_t \boldsymbol{f} - [\, \boldsymbol{B}_t^{\mathrm{int}} + \partial_{\boldsymbol{X}} \psi_t \,] \cdot \boldsymbol{V} - \mathrm{d}_{\boldsymbol{F}_{\mathrm{p}}} \psi_t : \mathrm{d}_t \boldsymbol{F}_{\mathrm{p}} \geq 0 \tag{8}$$

whereby $\boldsymbol{\pi}^{\mathrm{t}} = \mathrm{d}_{\boldsymbol{f}} \psi_t$ denotes the material motion first Piola–Kirchhoff stress. Comparing eqs.(3) and (8) ends up with the remarkable result

$$-\, [\, \boldsymbol{B}_t^{\mathrm{int}} + \partial_{\boldsymbol{X}} \psi_t \,] \cdot \boldsymbol{V} - \mathrm{d}_{\boldsymbol{F}_{\mathrm{p}}} \psi_t : \mathrm{d}_t \boldsymbol{F}_{\mathrm{p}} = -\, \det(\boldsymbol{f})\, \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : [\, \mathrm{d}_t \boldsymbol{F}_{\mathrm{p}} - \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} \cdot \boldsymbol{V} \,] \tag{9}$$

which we restate via

$$\det(\boldsymbol{F})\, \mathrm{d}_{\boldsymbol{F}_{\mathrm{p}}} \psi_t : \mathrm{d}_t \boldsymbol{F}_{\mathrm{p}} = \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \mathrm{d}_t \boldsymbol{F}_{\mathrm{p}} \quad \text{and} \quad \det(\boldsymbol{F})\, \boldsymbol{B}_t^{\mathrm{int}} = \boldsymbol{B}_0^{\mathrm{int}} = -\, \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} - \partial_{\boldsymbol{X}} \psi_0 \tag{10}$$

With these relations in hand, one observes that the material motion first Piola–Kirchhoff stress takes the format

$$\boldsymbol{\pi}^{\mathrm{t}} = \mathrm{d}_{\boldsymbol{f}} \psi_t = \mathrm{d}_{\boldsymbol{F}_{\mathrm{p}}} \psi_t : \mathrm{d}_{\boldsymbol{f}} \boldsymbol{F}_{\mathrm{p}} = \det(\boldsymbol{f})\, \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \mathrm{d}_{\boldsymbol{f}} \boldsymbol{F}_{\mathrm{p}} = \det(\boldsymbol{f})\, \boldsymbol{F}_{\mathrm{p}}^{\mathrm{t}} \cdot \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} \cdot \boldsymbol{F}^{\mathrm{t}} = \det(\boldsymbol{f})\, \boldsymbol{\Sigma}^{\mathrm{t}} \cdot \boldsymbol{F}^{\mathrm{t}} \tag{11}$$

For completeness, we finally highlight the material motion Cauchy or rather Eshelby stress $\boldsymbol{\Sigma}^{\mathrm{t}}$, which enters one of the balances of linear momentum in eq.(1), as well as the correlated source term

$$\boldsymbol{\Sigma}^{\mathrm{t}} = \det(\boldsymbol{F}_{\mathrm{p}})\, \boldsymbol{F}_{\mathrm{p}}^{\mathrm{t}} \cdot \mathrm{d}_{\boldsymbol{F}_{\mathrm{p}}} \psi_t = \boldsymbol{F}_{\mathrm{p}}^{\mathrm{t}} \cdot \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} \quad \text{and} \quad \boldsymbol{B}_0 = -\, \boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{b}_0 -\, \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} - \partial_{\boldsymbol{X}} \psi_0 \tag{12}$$

Remark 1 Please note that the self forces in eq.(3) due to $\partial_{\boldsymbol{x}} \psi_t$ are neglected since the free Helmholtz energy does not depend (explicitely) on $\boldsymbol{x}$. In fact, this is a consequence of invariance under superposed (spatial, orientation preserving) Euclidian transformations, i.e. $\psi_{\mathrm{p}}(\boldsymbol{\varphi}, \mathrm{D}_{\boldsymbol{X}} \boldsymbol{\varphi}, \ldots) = \psi_{\mathrm{p}}(\boldsymbol{\varphi}', \mathrm{D}_{\boldsymbol{X}} \boldsymbol{\varphi}', \ldots)$ with $\boldsymbol{\varphi}' = \boldsymbol{x}' = \boldsymbol{Q} \cdot \boldsymbol{x} + \boldsymbol{c}$ and $\boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{Q}^{-1}$.

Remark 2 An alternative derivation of eq.(12) is based on the classical representation of linear momentum, $\nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Pi}^{\mathrm{t}} + \boldsymbol{b}_0 = \boldsymbol{0}$, and the relations $-\, \boldsymbol{F}^{\mathrm{t}} \cdot [\, \nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Pi}^{\mathrm{t}} \,] = \boldsymbol{\Pi}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F} - \nabla_{\boldsymbol{X}} \cdot (\, \boldsymbol{F}^{\mathrm{t}} \cdot \boldsymbol{\Pi}^{\mathrm{t}} \,)$, with $\nabla_{\boldsymbol{X}} \boldsymbol{F}^{\mathrm{t}} : \boldsymbol{\Pi}^{\mathrm{t}} = \boldsymbol{\Pi}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F}$, and $\nabla_{\boldsymbol{X}} \cdot (\, \psi_0\, \boldsymbol{I} \,) = \boldsymbol{\Pi}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F} + \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} + \partial_{\boldsymbol{X}} \psi_0$, respectively.

Remark 3 The contribution to the internal material force stemming from either the irreversible or reversible part of the total deformation gradient allow representation in terms of, e.g., the material motion Cauchy or rather Eshelby stress, i.e $\boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} = [\, \boldsymbol{f}_{\mathrm{p}}^{\mathrm{t}} \cdot \boldsymbol{\Sigma}^{\mathrm{t}} \,] : \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} = \boldsymbol{\Sigma}^{\mathrm{t}} : [\, \boldsymbol{f}_{\mathrm{p}} \cdot \nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}} \,] = \boldsymbol{\Sigma}^{\mathrm{t}} : \boldsymbol{\Gamma}_{\mathrm{p}}$. Apparently, the skew part of $\nabla_{\boldsymbol{X}} \boldsymbol{F}_{\mathrm{p}}$ or $\boldsymbol{\Gamma}_{\mathrm{p}}$, respectively, characterize the dislocation density.

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