CONSTITUTIVE EQUATIONS OF MESOELASTIC DEFORMATION

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Summary

We study deformations of mesoelastic materials that display different types of imperfections with a typical size of 1μm. Stress-strain relationships of these materials depend on the processing history and exhibit common behaviour, including non-linearity, hysteresis, etc. We focus our study on the continuous distribution of singularities in the deformation field, which are described in terms of dislocation densities and fluxes. We define the mass mesodensity tensor and deduce the constitutive relationship between the dislocation current and the linear mesomomentum. Based on the modification of Peach-Koehler formula we propose the constitutive relationship between the line mesostress tensor and the dislocation density. These constitutive relationships allow us to model stresses in mesoelastic materials.

INTRODUCTION

Deformation stress in elastic materials may appear not only by the actions of external forces on them but also because of singularities of internal structure. This paper represents our study of the role of dislocations as a source of stress fields. Experimental studies of deformations of elastic materials with imperfections of a typical size of 1μm indicate that their stress-strain relationships depend on the processing history and show a common behaviour, exhibiting non-linearity, hysteresis, etc [1]. We assume that these imperfections play a significant role in the development of singularities of deformation gradient $\boldsymbol{\beta}$.

The continuum of dislocations is described in term of the dislocation density tensor $\boldsymbol{\alpha} = \boldsymbol{\nabla} \times \boldsymbol{\beta}$. The balance equation of the Burgers vector in defect dynamics $\boldsymbol{\nabla \times J} + \boldsymbol{\partial_t \alpha} = \mathbf{0}$ relates the dislocation current (the flux density) **J** and the dislocation density $\boldsymbol{\alpha}$ [2].

Peach and Koehler's study of the action of an elastic stress field on a dislocation led to the discovery that the force acting on a unit length of a dislocation line has the following form: $f^i = \varepsilon^{ikl} \tau_k S_{lm} b_m$, where ε^{ikl} is antisymmetric in the k and l indices, τ_k is the tangent vector to the dislocation loop, **S** is the elastic stress, and **b** is the Burgers vector [3].

DYNAMICS

The main point of our consideration is to reveal a stress response to the continuum of dislocations. We associate this response with, as we call it, mesostress, which arises in an elastic body in addition to the elastic stress.

Dislocation Mass and Linear Momentum of Dislocation

We evaluate the mass density vector of a dislocation as the following:

$$\rho^i = \oint_L \rho_{ref} \beta^{ij} dX_j ,$$

where ρ_{ref} is the material mass density in the reference configuration.

Also, we use the Kosevich formula [1] for the dislocation current

$$\mathrm{J_{ik}} = \varepsilon_{ilm} \tau^l \mathrm{b_k} \mathrm{V^m} \delta(\xi), \tag{1}$$

where **V** is the velocity at a given point on the dislocation line, δ(ξ) is a two-dimensional delta function, and ξ is a two-dimensional position vector from a given point on the dislocation axis in a plane perpendicular to the tangent vector τ.

In order to account for the relations between inertia of the dislocation movement and the dislocation mass, we define

$$\gamma^{\mathrm{mn}} = \rho^{\mathrm{n}} \mathrm{V^m} \delta(\xi)$$

Then the Kosevich formula (1) takes the following form:

$$\mathrm{J_{ik}} = \varepsilon_{\mathrm{ilm}} \frac{\alpha^{\mathrm{l}}{}_k}{\rho_n} \gamma^{\mathrm{mn}} = L_{ikmn} \gamma^{\mathrm{mn}} \tag{2}$$

Continuous distribution of dislocations

In accordance with our point of view, mesoelastic material incorporates a continuously distributed network of dislocations. For this material we propose the following generalization of (1):

$$\boldsymbol{\gamma} = \boldsymbol{\rho} \mathbf{J}$$

Here we call the tensor γ the surface linear mesomomentum and $\boldsymbol{\rho} = \mathbf{L}^{-1}$ as the dislocation mass mesodensity tensor in the continuum.

The mesomomentum of inertia **P** for any region U on a body B can be written as flux over closed surface ∂ R bounding this volume

$$\mathbf{P}=\iint_{\partial \mathbf{U}} \gamma \cdot \mathbf{dA}=\iiint_{\mathbf{U}} \mathbf{p}\mathbf{dV}$$

Here vector p we call linear mesomomentum. The balance of mesomomentum for smooth functions and for any portion U of a body B provides conditions for the formula $\mathbf{p}=\nabla \cdot \boldsymbol{\gamma}$.

Line Mesostress

We consider a line with a unit normal cross section area. We call a force **P** acting along the line a mesoforce and define the line mesotraction to be **f =dP/dl**. For the mesocontinuum of these lines we postulate the existence of the line mesostress tensor **I= I(X,t)** with the following property: $\mathbf{f}=\mathbf{I}\cdot\boldsymbol{\tau}$ and define surface mesostress tensor $\mathbf{C}= \nabla \times \mathbf{I}$ [4]. In the following consideration we regard these lines as dislocation lines in elastic body.

Line Mesostress for Continuum of Dislocations

We choose the following generalization of Peach Koehler formula for the continuum of dislocations: $\mathbf{I}=\mathbf{K}(\mathbf{S})\cdot\boldsymbol{\alpha}$

MESOELASTIC MODEL

Here we assume that the mesostresses play the main role in the response to the material deformation. For this case we combine the following mesoelasticity system:

$$\nabla \times \mathbf{I}+\partial_t \gamma = \mathbf{C}, \;\; \nabla \cdot \gamma=\mathbf{p}$$

$$\nabla \times \mathbf{J}+\partial_t \alpha = \mathbf{0}, \;\; \nabla \cdot \alpha=\mathbf{0}$$

with the constitutive expressions:

$$\mathbf{I}=\mathbf{K}(\mathbf{S})\cdot\boldsymbol{\alpha} \text{ and } \boldsymbol{\gamma}=\boldsymbol{\rho}(\boldsymbol{\alpha})\cdot\mathbf{J}$$

Statics

We consider the statics of an irregular structure as a steady state. Thus, we assume that **H=0, p=0** and **γ=0**. Then the system of equations is reduced to the following form:

$$\nabla \times (\mathbf{K}(\mathbf{S})\cdot\boldsymbol{\alpha}) = \mathbf{C} \qquad \nabla \cdot \boldsymbol{\alpha} = \mathbf{0}$$

Taking into account that $\alpha=\nabla \times ß$ we transform the system above, in the case of constant **K(S)** and $\nabla \cdot ß=\mathbf{0}$, to the integral equation:

$$ß(\mathbf{y})=\frac{1}{4\pi}\mathbf{K}^{-1}(\mathbf{S})\int_{\mathbf{B}} \mathbf{G}(\mathbf{x},\mathbf{y})\mathbf{C}\mathbf{dV}$$

If we assume that an unloading of the stressed mesoelastic body B is pure elastic then the mesoelastic stress C is responsible for the existence of the residual stress in that body.

CONCLUSIONS

The presented constitutive theory for material deformations describes the constitutive relationship between the dislocation current and the linear mesomomentum. This relationship is based on the generalization of the Kosewich formula. Based on the generalization of the Peach-Koehler formula, we propose the constitutive relation between the line mesostress tensor and the strain singularities density. These constitutive relationships have allowed us to model the stress of the mesoelastic materials.

References

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