CHAOTIC ATTRACTORS WITH LONG REGULAR SEQUENCES

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Summary This paper presents examples of chaotic motions in non-smooth mechanical systems affected by dry friction. The chaotic attractors are composed of zones characterized by very different rates of divergence of nearby orbits. The mechanical system generates one-dimensional maps the orbits of which seem to exhibit sensitive dependence on initial conditions only in an extremely small set of their field of definition. The Lyapunov exponent of the map is computed to characterise the steady state motions.

INTRODUCTION

Chaotic motions are characterised by sensitive dependence on initial conditions, that means that nearby orbits belonging to a chaotic attractor (and corresponding to nearly identical states) will soon behave differently [1]. Rates of divergence or convergence of nearby orbits are different in different zones of an attractor. Lyapunov exponents are a quantitative measure of the *average* exponential rates; they give information on the whole attractor and therefore do not provide any clear information on the zones where divergence rates are higher or lower. This paper aims at presenting an example of chaotic attractor in which it is possible to clearly identify distinct zones with different divergence properties. The idea of distinguishing zones in chaotic attractors with different divergence properties is not commonly pursued in the scientific literature and represents an element of novelty of the present paper.
The dynamical system under investigation is a mechanical system subjected to elastic and dry friction forces which exhibits many features typical of non-smooth dynamical systems as chaotic motions, non-smooth transitions and non-smooth bifurcations [2, 3].

THE MECHANICAL SYSTEM AND ITS MAP

The mechanical system

The mechanical system investigated in this paper is shown in figure 1. It is composed of two blocks supported by a belt which moves with constant velocity V_{dr}. Elastic springs couple the blocks and connect them to a fixed support. Dry friction forces act between the blocks and the belt and they can be static, when the blocks ride on the belt, or kinetic, when the blocks slip with respect to the belt. The kinetic friction characteristic F_k is assumed to be a continuous, single-valued decreasing function of the velocity of the block relative to the belt [3]:

$$F_k(\dot{X}-V_{dr}) = \begin{cases} \dfrac{F_s}{1-\gamma(\dot{X}-V_{dr})} & \text{if } V_{dr} > \dot{X} \\ -\dfrac{F_s}{1+\gamma(\dot{X}-V_{dr})} & \text{if } V_{dr} < \dot{X} \end{cases}$$

Where F_s, the maximum static friction force, is unity for block 1 and 0.9 for block 2. The blocks have the same mass but the system is not symmetric because of the difference in the maximum static friction forces. The following parameters completely define the mechanical system: $m=1$, $k=1$, $k_{12}=1.2$, $\gamma=3.0$, $V_{dr}=10^{-7}$.

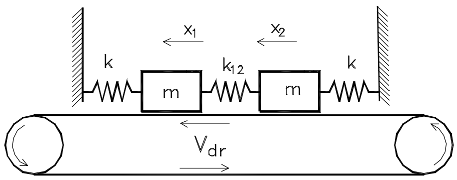

Figure 1

One-dimensional map

The 2-block stick-slip system of figure 1 can generate a one-dimensional map [3]. When both blocks are riding on the belt (such a phase of motion will be called global stick phase, g.s.p.) the relative displacement between the two blocks is fixed; therefore the g.s.p.'s of the system may be characterised by the constant value of a variable d:

$$d=X_2-X_1$$

The g.s.p. will finish where one of the two blocks starts slipping. Then the motion of the slipping block may trigger a new slipping phase also for the other block but eventually, if V_{dr} is sufficiently small, both blocks will reach a configuration in which they ride simultaneously on the belt. The new g.s.p., in general, will be characterised by a value of the relative displacement d different from the one assumed during the previous g.s.p. In this way a motion of the system generates an infinite sequence of values of the variable $d=d_1, d_2, d_3, \ldots$, which can be interpreted as a map expressing d_{k+1} as a function of d_k:

$$d_{k+1}=f(d_k)$$

The 1-dimensional map is single valued because a given value of the variable d univocally determines the initial conditions at which one of the blocks will start slipping. The one-dimensional map will be used to describe in a compact way the dynamics of the two-block model; in particular it will be used to compute the main Lyapunov exponent of the system according to [4].

RESULTS

Figure 2 shows a portion of the 1-d map generated by the system of figure 1 and two enlarged views. Note that the scales of the figures are different so that the main properties of the maps can be appreciated. A considerable portion of the map seems to be parallel to the line $d_{k+1}=d_k$ and very close to it. When the orbit of the map is captured within that region, figure 2 – enlarged view 'a', a large number of 'regular' iterations of the map takes place. This sequence of regular iterations is interrupted when the orbit of the map reaches the zone where the slope of the map changes suddenly and in a discontinuous way ('chaotic' zone), figure 2 – enlarged view 'b'. Then in a very small number of iterations the orbit is pushed back in the 'regular' zone. The attractor is therefore composed of two qualitatively different zones: a 'regular zone', the branch 'parallel' to the line $d_{k+1}=d_k$, where neighbouring orbits do not separate with high rate, and a 'chaotic zone', the branch with steep slopes, which separates nearby orbits and send them back in different positions of the 'regular zone'. Note that, in the enlarged view 'b', the horizontal scale is magnified by 100 and the almost horizontal branch to the right is the most left part of the 'regular zone'.
The dynamics of the map, and therefore of the underlying mechanical system, can be characterised by the Lyapunov exponent of the 1-d map [4]:

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \Sigma \log \left| f'(d_k) \right|$$

where $f'(d_k)$ is the derivative of the map, that can be approximately computed with suitable finite difference techniques. The limit in the above expression can be evaluated by integrating the dynamics of the map for a very large number of iterations [4]. Figure 3 shows how the numerical computation of the Lyapunov exponent 'converges' to a value bigger than zero. The enlarged view of figure 3 shows that the diagram of the value of the exponent versus the number of iterations has a saw-tooth shape due to the alternate occurrence of 'regular' and 'chaotic' zones. Therefore the computation of the Lyapunov exponent confirms the idea that of an attractor composed of zones characterised by clearly different dynamic behaviours. Several algorithmic parameters affect the computed value of the Lyapunov exponent which, in the range between 400000 and 1 million iterations, fluctuates between 0.0022 and 0.0028.

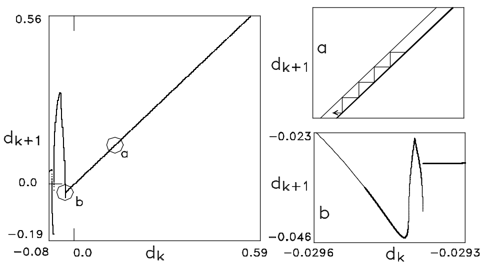

Figure 2

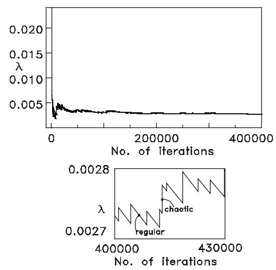

Figure 3

CONCLUSIONS

This paper describes a chaotic attractor in which the zone of high divergence rate of nearby orbits (chaotic zone) is clearly distinct from another zone in which the divergence rate is much lower (regular zone). In the regular zone the dynamics of the system is composed of long sequences of similar iterations whereas an orbit can only stay in the chaotic zone for a small number of iterations. The numerical calculation of the Lyapunov exponent confirms that the motion is composed of long regular phases interrupted by short chaotic phases. Some caution has to be used when examining the numerical results.

References

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