The effect of smoothing on bifurcation and chaos computations in non-smooth mechanics

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In rigid body mechanics involving impacts, backlash or friction, mathematical models of these phenomena involve functions that are discontinuous. These non-smooth mechanical systems have been the subject of considerable interest of late. The review [1] is an excellent introduction to this field. See also [2].

These systems can be described by a set of equations of the form

$$\dot{x} = f(x, t, \mu) \tag{1}$$

where t is time, $x \in \mathbb{R}^n$ is a state vector, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a non-smooth function and $\mu \in \mathbb{R}^p$ is a vector of parameters. Systems can be autonomous or non-autonomous. The function f is smooth in countably many (finite or infinite) regions of phase space, but can have a different functional form in each region.

Not only are these systems of significant importance to engineers but also they present difficult challenges to mathematicians. *Border-collision* or *C*-bifurcations occur when the system steady state (or Ω limit set) intersects a region boundary under parameter variation. See [3] for a further details.

When numerical computations of the governing equations are carried out, especially when detection of bifurcation or chaos is important, a smoothed version of the dynamics is often used.

For example the step function $S(x)$ defined by

$$S(x) = \begin{cases} 1 & \text{when } x > 0, \\ 0 & \text{when } x = 0, \\ -1 & \text{when } x < 0. \end{cases} \tag{2}$$

occurs frequently in this context and can be approximated by $S(x) = \tanh Kx$. The error in this approximation decreases as $K \to \infty$. The dynamics that persist in this limit are taken (numerically at least) to be those that are present when (2) is the chosen form for $S(x)$.

In this paper we show how smoothing always introduces spurious solutions into some part of parameter space. We also show how the location of the bifurcation point itself varies according the way in which the smoothing is carried out. In particular we show how to correct for this variation.

We take the simple generic example treated in [3]

$$x^{n+1} = \begin{cases} \alpha x^n - \mu, & x^n > 0 \\ \beta x^n - \mu, & x^n < 0 \end{cases} \tag{3}$$

where for definiteness we take $\alpha > 0$ and $\beta < 0$.

We replace (3) with

$$x^{n+1} = \frac{1}{2}x^n[(\alpha + \beta) + (\alpha - \beta)\tanh Kx^n] - \mu \tag{4}$$

which tends to (3) as $K \to \infty$.

Fixed points x^* of (4) now satisfy

$$\frac{\mu}{x^*} = [\frac{1}{2}(\alpha + \beta) - 1] + \frac{1}{2}(\alpha - \beta)\tanh Kx^* \tag{5}$$

Graphical means are then used to show that (5) always has two extra roots in some part of parameter space when K is finite. These spurious solutions are not part of the original dynamics of the non-smooth system.

The tanh function in (4) is then expanded as a Taylor series and the bifurcation points determined analytically for small K. These are given by

$$x^* = (\beta - \alpha - 2)/2K(\alpha + \beta) \tag{6}$$

These differ from zero for all finite K. This error can be corrected for by replacing (4) by

$$x^{n+1} = \frac{1}{2}x^n[(\alpha + \beta) + (\alpha - \beta)\tanh K(x^n - x_0)] - \mu \tag{7}$$

for a suitable choice of x_0.

Finally we conclude by showing how different border-collision or C-bifurcations are different limits of smooth bifurcations. This is done by computing the bifurcation diagrams for (7) for different (fixed) values of α, β and varying K. The particular type of bifurcation found at these values of α, β is then shown to be the limit of a smooth bifurcation sequence for finite K.

References

[1] Popp, K. *Non-smooth mechanical systems* J. Appl. Math. Mech., **64**, 2000, 765–772.

[2] Babitsky, V.I. and Krupenin, V.L. *Vibrations of Strongly Nonlinear Systems* Springer Verlag, Berlin, 2001.

[3] Di Bernardo, M., Feigin, M., Hogan, S.J. and Homer, M.E. *Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems* Chaos, Solitons and Fractals **10**, 1999, 1881–1908.