A LONG-WAVELENGTH MODEL OF VISCOUS ENTRAINMENT

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Summary When a large air bubble rises in syrup, it often leaves behind a thin trailing tendril in which air is entrained into the syrup. This is a familiar example of how viscous entrainment can create a long and slender structure on a liquid surface. We derive a simplified model of viscous entrainment in the limit when the entrained fluid is far less viscous than the entraining fluid. Results suggest there exists a class of macroscopic conditions which allow local, scale-invariant entrainment dynamics, thus raising the possibility of that infinitely thin liquid spouts in the continuum model are realizable in practice.

INTRODUCTION

If you turn a half-empty jar of honey upside down, (with the lid closed, of course), you will see a large air bubble form and rise slowly through the honey. If the jar is large, this air bubble will rise for a considerable distance before reaching the top. As it rises, the trailing end of the bubble becomes tapered and eventually leaves a tendril of air behind. Similar phenomena occur when a less viscous liquid is entrained by flow in a more viscous fluid, as relevant in a variety of physical contexts, such as oil recovery, mixing in chemical reactors, and has been proposed as a possible mechanism for the formation and the stability of localized upwelling of hot materials from deep mantle, i.e. hot-spots [1].
Here we ask what determines the volume flux of fluid left behind when a moving liquid drop is deformed sufficiently to just begin to leak at its trailing edge? This question is inspired by a series of experiments on viscous withdrawal by Cohen & Nagel [2]. We are particularly interested in the possibility that the entrainment flux goes to 0 continuously at a critical value of the imposed withdrawal rate, since it corresponds to the steady-state interface undergoing a continuous evolution from a hump to a spout as the withdrawal rate is tuned across the critical value. Fundamentally, the existence of such a continuous transition in the steady-state shape is unusual, because the two fluid layers are immiscible and therefore the interface has a finite surface tension. As a consequence, there exists a Laplace pressure across the interface which is proportional to a product of surface tension and mean curvature. If the curvature becomes infinite at a point, the Laplace pressure also becomes infinite, and stresses in the vicinity of the singularity balance if and only if the viscous stresses also diverge. This corresponds to the formation of a steady-state singularity.

PROBLEM FORMULATION

We design a simple, experimentally realizable model problem (figure 1a) to investigate the possible formation of a steady-state singularity. We consider the entrainment of a thin cylindrical spout of less viscous fluid from a nozzle by an imposed axial straining flow in an exterior fluid $\mathbf{U}_\infty(r,\theta,z) = \left(-\frac{Er}{2},\ 0,\ Ez\right)$. Inertial effects are taken to be negligible everywhere. The nozzle fluid is taken to be far less viscous than the exterior fluid so that the spout surface is well approximated by a surface of zero tangential stress. Most importantly, we assume that the spout profile is long and slender everywhere and look for a long-wavelength model.

In deriving a long-wavelength equation for the case where the interior viscosity $\mu_{\rm int}$ is much less than the exterior viscosity $\mu_{\rm ext}$, we follow Taylor [3, 4] in modeling the effect of the free-stress surface on the exterior flow as a radial flow, $q(z)/(2\pi r)$, corresponding to a line of point sources which are situated along the drop centerline. The strength of the volume flux, $q(z)$, is determined by the kinematic condition. Since Taylor's model was for the steady-state shape of a drop in an axial straining flow, the entrainment flux $Q_{\rm spout}$ is always 0 for his case. Moreover global volume conservation for the drop requires that the interior pressure $P_{\rm int}$ for Taylor's problem satisfy an integral constraint. In our case, we assume the nozzle is connected to an extremely large reservoir, so that $P_{\rm int}$ is determined by local stresses. In steady-state the spout profile $R(z)$ satisfies the nonlinear ordinary differential equation

$$Q_{\rm spout} = 2\pi \left\{ \frac{EzR^2(z)}{2} - \frac{R^4(z)}{16\mu_{\rm int}} \frac{\mathrm{d}}{\mathrm{d}z} \left[2\mu_{\rm ext}E \left(1 + \frac{z}{R(z)}\frac{\mathrm{d}R}{\mathrm{d}z}\right) + \gamma\left(\frac{1}{R} - \frac{\mathrm{d}^2R}{\mathrm{d}z^2}\right)\right]\right\} \tag{1}$$

where $Q_{\rm spout}$ is an unknown quantity and γ the surface tension. The idea is that for a given imposed strain rate E only a specific value of $Q_{\rm spout}$ can satisfy upstream boundary conditions at the nozzle and the condition that a spout be entrained downstream, i.e. that the equation for $R(z)$ supports a solution which extends to positive infinity. Equation (1) can be recast into a simpler, nondimensional form by the nondimensionalizations $\hat{r} = R_N$, $\hat{z} = \frac{\hat{r}}{2\sqrt{\lambda}}$,

$\hat{P} = \frac{\gamma}{\hat{r}}$ and $\hat{U} = \frac{\gamma}{\mu_{\text{ext}}}$ where R_N is the nozzle radius. In our model problem the entrainment dynamics is characterized by two dimensionless parameters, the viscosity contrast $\lambda = \mu_{\text{int}}/\mu_{\text{ext}}$, assumed to be far less than 1, and related only to the material properties and a capillary number $Ca = (2\mu_{\text{ext}} E R_N)/\gamma$ where $Ca \gg 1$ corresponds to the limit of strong flow and weak surface tension. From this point on all the variables are dimensionless unless otherwise specified.

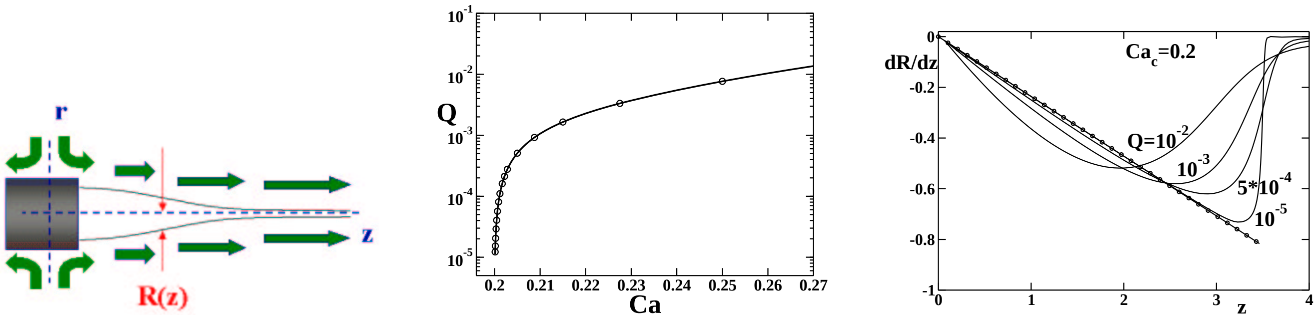

FIG. 1: a) An imposed axial straining flow entrains a steady-state spout from a fluid filled nozzle connected to a large reservoir. b) Steady-state entrainment flux Q versus Ca for nozzle boundary conditions with $Ca_c = 0.2$. The viscosity contrast λ is 1/80. c) Slope profile of full solutions for $Ca_c = 0.2$ at $Q = 10^{-2}$, 10^{-3}, $5 \cdot 10^{-4}$, and $Q_{\min} = 10^{-6}$. The solid line with circles corresponds to $\mathrm{d}R_D/\mathrm{d}z$.

RESULTS

There exists a special class of nozzle boundary conditions which correspond to continuous entrainment transitions in (1). Fundamentally this is because the equation supports a family of exact solutions

$$R_D(z) = 1 - \left(\frac{z}{z_D}\right)^2 \qquad z_D = 2\sqrt{\frac{1}{2Ca_c} - 1} \tag{2}$$

which correspond to $Q = 0$ solutions which end in a cone. From similarity solutions of (1), we know that such conical profiles correspond precisely to the upstream profiles needed for an infinitely thin liquid spout to be entrained [5]. Thus specifying boundary conditions at the nozzle consistent with the existence of (2) at $Ca = Ca_c$ makes it possible for the entrainment volume flux Q to vanish continuously as $Ca \to Ca_c$. Numerical solutions of (1) show indeed this continuous transition, as illustrated by figure 1b. The entrainment flux Q scales linearly with $Ca - Ca_c$ over several decades in Q. For more details, see [5]. Figure 1c shows that the upstream spout profiles at finite, but small Q, converge towards (2) as Q becomes small. In conclusion, full solutions of the long-wavelength model and similarity solutions suggest that liquid spouts of atomic thickness and macroscopic extent can indeed be created at a viscous entrainment transition when appropriate upstream boundary conditions are imposed.

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