# ON THE RAYLEIGH-BÉNARD PROBLEM IN THE CONTINUUM LIMIT

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*Summary* The transition to convection in the Rayleigh-Bénard problem at small Knudsen numbers is studied via a linear stability analysis of the compressible 'slip-flow' problem. No restrictions are imposed on the magnitudes of temperature difference and compressibility-induced density variations. Comparison of the results with existing DSMC and continuum non-linear simulations demonstrates that the present analysis correctly predicts the boundaries of the convection domain. This offers the linear temporal stability analysis as a viable means of studying the effects of the various parameters on the onset of convection.

## INTRODUCTION

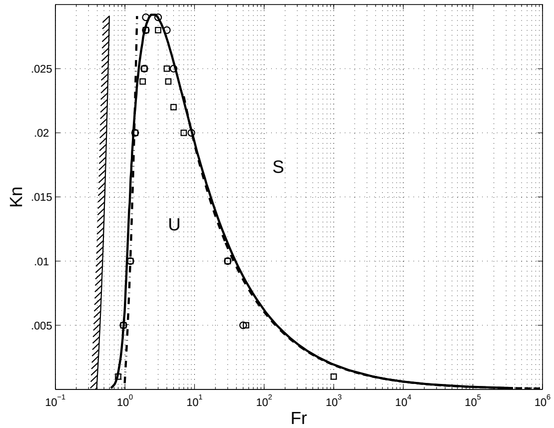
The onset of convection in an initially quiescent fluid confined between parallel horizontal walls and heated from below is a classical problem in hydrodynamic stability theory (Chandrasekhar 1961). This problem has been investigated extensively within the framework of the Boussinesq approximation where fluid-density variations are only considered in the buoyancy term of the equation of motion. This approximation is based on the assumptions that relative temperature differences and density variations owing to compressibility are both small. Each of the relatively few analyses which have hitherto addressed the corresponding compressible-flow problem have relaxed either one but not both of these assumptions. Thus, Fröhlich, Laure and Peyret (1992) studied the effects of large temperature differences while retaining the assumption that compressibility-induced density variations were negligible. Bormann (2001), among others, considered compressibility effects assuming small temperature differences and hence constant transport (i.e. viscosity and thermal conductivity) coefficients. In both cases the results indicate that the compressible-fluid system is less unstable than the comparable predictions based on the Boussinesq approximation. To the best of our knowledge, no comprehensive treatment of compressible-fluid stability problem exists in the literature even within the context of continuum gas-dynamics. Thus, addressing this problem without a priori restricting the relative temperature differences or compressibility-induced density variations is one of the main objectives of the present contribution.

The Rayleigh-Bénard (RB) problem for rarefied gases has been studied in recent years principally by means of the Direct Simulation Monte Carlo (DSMC) method (see Stefanov, Roussinov & Cercignani 2002, hereafter referred to as SRC, and references cited therein). These numerical simulations follow the evolution of the macroscopic (hydrodynamic) fields through their terminal states. The results demonstrate that significant convection only occurs in the continuum limit of small $O(10)^{-2}$ Knudsen numbers. However, 'noisy' elements inherent in DSMC make it difficult to characterize the final states, particularly for parameters combinations in the vicinity of the transition to convection. Furthermore, these simulations become extremely time consuming in the continuum limit, which obstruct an accurate delineation of the domain of instability. An alternative approximate analysis of the onset of RB convection at small $Kn$ numbers thus constitutes our main goal.

## PROBLEM STATEMENT AND ANALYSIS

In view of the above we focus on the continuum limit. The commonly-accepted model in this limit is the 'slip-flow' problem consisting of the familiar continuum (i.e. continuity, Navier-Stokes and energy) equations in conjunction with first-order velocity-slip and temperature-jump conditions at the boundaries (which represent the effects of slight rarefaction). The parameters governing this problem are the Knudsen ($Kn$) number representing the ratio of the microscopic (the mean free path) and macroscopic (the distance between the walls) scales, the Froude ($Fr$) number characterizing the relative magnitude of the thermal-inertial and gravitational effects, and $R_T$, the ratio of the upper (cold)- and lower (hot)-wall temperatures. This is in marked contrast to the above-mentioned Boussinesq problem which is exclusively governed by the Rayleigh number (which in the present notation is inversely proportional to $FrKn^2$). No assumptions restricting the respective magnitudes of the temperature differences and compressibility effects are made. This allows us to consider the relevant (according to existing DSMC results) domain of parameters.

In a simple monatomic gas consisting of hard-sphere molecules the viscosity and thermal conductivity are both proportional to the square root of the absolute temperature. For this law of molecular interaction we make use of the closed-form expressions obtained by SRC for the reference 'pure conduction' state. A linear temporal stability analysis is performed assuming small three-dimensional perturbations which are spatially harmonic in planes parallel to the walls and linearizing the governing equations and boundary conditions about the above reference state. However, owing to symmetry , the resulting problem is essentially two-dimensional in a vertical plane parallel to the wave-number vector. The dispersion relation providing the growth rate of perturbations $\omega$ as a function of their wave-number $k$ and the above set of parameters ($Kn$, $Fr$ and $R_T$) is calculated by transforming the system of ordinary differential perturbation equations into an algebraic eigenvalue problem via application of the Chebyshev collocation method.




Division of the plane of parameters $(Fr, Kn)$ into unstable $(U)$ and stable $(S)$ domains for $R_T = 0.1$ according to the present theory (solid line) together with DSMC (circles) and continuum (squares) results of SRC. Also marked are the large-$Fr$ asymptote $Ra \approx 1773$ (dashed line), the initial appearance of non-monotonical density distribution (dash-dotted line) and the necessary condition for the onset of convection (cross-hatched line).

## RESULTS AND DISCUSSION

Throughout the entire domain of parameters our calculations invariably yield real-valued $\omega$. Accordingly, the onset of convection takes place via "exchange of stabilities" (Chandrasekhar 1961), i.e. $\omega = 0$. In the following we focus on a temperature ratio $R_T = 0.1$ so as to facilitate comparison with the results of SRC.

The solid line in the attached figure separates the plane of parameters $(Fr, Kn)$ into respective domains of stable ($S$), $\omega < 0$, and unstable ($U$), $\omega > 0$, response. Also presented are the corresponding DSMC (circles) and continuum finite-difference (squares) results of SRC, the large-$Fr$ asymptote ($FrKn^2 \approx 3.65 \times 10^{-3}$, dashed line), the dash-dotted line marking the locus of states where non-monotonical density distributions initially appear in the pure-conduction reference state and the cross-hatched line corresponding to the necessary condition for instability related to compressibility-induced density variations (see the discussion below).

For all $Kn > 0$ the convection domain is confined to a finite interval of Froude numbers. The extent of this interval is rapidly diminishing with increasing $Kn$, vanishing entirely for $Kn >\approx 0.029$. When $Kn \approx< 0.01$ (and $Fr \gg 1$), the right branch of the $U$-domain boundary is approaching the dashed line. In this limit compressibility effects are negligible and the asymptote corresponds to a constant Rayleigh number (based on the arithmetic mean of wall temperatures) $Ra \approx 1773$. This coincidence with a critical value of $Ra$ which is larger than the Boussinesq value ($\approx 1708$) is in qualitative agreement with the results of Frölich *et al.* (1992).

Considering the left branch of the $U$-domain boundary, we note that for all $Kn <\approx 0.02$ this boundary is disposed to the left of the dash-dotted line. Indeed, transition to convection may take place in a compressible fluid even when the fluid density in the reference state is monotonically decreasing (Bormann 2001). Thus, convection may set in provided that adiabatic expansion of a fluid element rising through the reference hydrostatic pressure field reduces its density below the ambient reference density (Landau and Lifshitz 1959). The cross-hatched line in the figure presents the locus of points where this condition is initially satisfied. The analysis leading to this criterion does not consider the retarding effects of fluid viscosity and heat conductivity, hence it is in fact only a necessary condition for the onset of convection.

In view of the vastly different methods of calculation (a linearized eigenvalue problem as opposed to a non-linear initial value problem), the close agreement between the present results and those of SRC is gratifying. The largest differences between the respective results appear at the lower portion of the right branch of the boundary of the convection domain and may be attributed to non-linear (hysteresis) phenomena which are obviously absent from the present linear analysis. Traversing the rest of the boundary of the $U$ domain we observe that the differences between the present line and the results of SRC are comparable to the differences between their various schemes. The linear analysis thus offers a useful alternative in studying the effects of various parameters and models of molecular interaction on the onset of convection.

## References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