Commutator-errors in large-eddy simulation of turbulent flow

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Abstract

Commutator-errors in large-eddy simulation of incompressible turbulent flow arise from the application of non-uniform filters to the continuity – and Navier-Stokes equations. As a consequence, the filtered velocity field is no longer solenoidal. For non-uniform, high-order filters the order of magnitude of the commutator-errors is shown to be the same as that of the turbulent stress fluxes. Consequently, one can not reduce the size of the commutator-errors independently of the turbulent stress terms by some judicious construction of the filter operator. Independent control over the commutator-errors compared to the turbulent stress fluxes can only be obtained by appropriately restricting the spatial variations of the filter-width and filter-skewness. For situations in which the dynamical consequences of the commutator-errors are significant, e.g., near solid boundaries, explicit similarity modeling for the commutator-errors is proposed, including the application of Leray regularization. The performance of this commutator-error parameterization is illustrated for the one-dimensional Burgers equation. The Leray approach captures the filtered flow with higher accuracy than conventional similarity modeling, which is particularly relevant for large filter-width variations.

In large-eddy simulation (LES) of turbulence one aims to predict the primary features of an unsteady flow without explicitly resolving all dynamically relevant length-scales. The modeling of turbulent flow in large-eddy simulation starts from the introduction of a spatial, low-pass filter with externally specified filter-width Δ. This allows to locally distinguish flow-features with a length-scale larger than Δ from flow-features with length-scale smaller than Δ. In the LES context, the former are referred to as 'resolved' while the latter class of flow-structures is identified as 'subgrid' or 'sub-filter'. During a simulation, the time-dependent resolved scales are explicitly calculated while the dynamic effects of the subgrid scales on the evolution of the resolved scales is represented through the introduction of an explicit 'subgrid' model. In the filtering approach to large-eddy simulation the spatial filter is traditionally assumed to be characterized by a single, spatially uniform operator, in particular, with a single value for Δ throughout the flow domain.

The desire to extend large-eddy simulation to flows in complex domains generally implies that one is confronted with strongly varying turbulence intensities within the flow-domain. In the filtering approach this can be accommodated using a filter operator with non-uniform filter-width. The use of such filters, however, complicates the subgrid closure problem through the appearance of additional commutator-errors. These terms arise from the fact that non-uniform filters do not commutate with differentiation, e.g., $\overline{\partial_x u} \neq \partial_x \overline{u}$ where ∂_x denotes differentiation of the solution u with respect to x and the overline indicates the filter operation. In our contribution we will *(i)* derive the complete, non-uniformly filtered equations and establish the magnitude of the additional commutator-error closure-terms relative to the fluxes arising from the turbulent stress tensor, *(ii)* identify under what conditions for the filter, the commutator-errors are likely to be dynamically relevant and *(iii)* for such situations, propose and illustrate explicit similarity modeling based on Leray regularization. These findings support the extension of the large-eddy approach to turbulence under realistic flow conditions and in complex flow domains. Some applications in three spatial dimensions will be included in the final contribution.

There have been considerable developments related to the application of such non-uniform filters in recent literature. These developments are primarily associated with the construction of specific high-order filters. For such general 'N-th order' filters the magnitude of commutator-errors can be controlled to some degree. Specifically, in literature it is emphasized that the size of the commutator-errors can be sufficiently reduced for these terms to become negligible, simply by appropriately increasing the order of the adopted filter. This impression is, however, somewhat over-simplifying matters since it is not fully complete. In the present paper we take a different approach to commutator-errors. In fact, it is shown that non-uniform filtering leads to commutator-errors which are of the same formal order of magnitude as the turbulent stress fluxes in the filtered Navier-Stokes equations. This holds for any higher-order filter and emphasizes the point that the magnitude of the commutator-errors can not be reduced *independently* of the turbulent stress fluxes by high-order filtering only; commutator-errors can not simply be 'filtered away' independently. Consequently, the question under which conditions commutator-errors are dynamically relevant, is still open.

Rather than controlling the size of the commutator-errors by increasing the order of the filter, the magnitude of these contributions can also be influenced by controlling the spatial variations in the filter properties. We sketch under what conditions the commutator-errors can be expected to be negligible relative to the turbulent stress fluxes, i.e., identify situations such that the commutator-errors do not require explicit closure. Conversely, if these conditions can not be satisfied, the introduction of an additional, explicit subgrid model for the commutator-errors is warranted. For this purpose, we present an alternative formulation for LES based on Leray regularization which allows to systematically develop a subgrid model for the commutator-errors. The explicit Leray modeling is compared to extended traditional similarity subgrid modeling.

To provide a modest illustration of explicit dynamical consequences of commutator-errors and their similarity models, the evolution of running 'ramp-cliff' waves in the viscous Burgers equation is studied. It is shown that the 'Leray-regularized' formulation provides a better representation of the non-uniformly filtered velocity field than the usual similarity model. In particular, the Leray approach provides a simultaneous capturing of turbulent stress fluxes as well as commutator-errors without increasing computational costs compared to the traditional case of uniform filters. The extension toward non-uniformly filtered turbulent flow in three dimensions is subject of ongoing research.

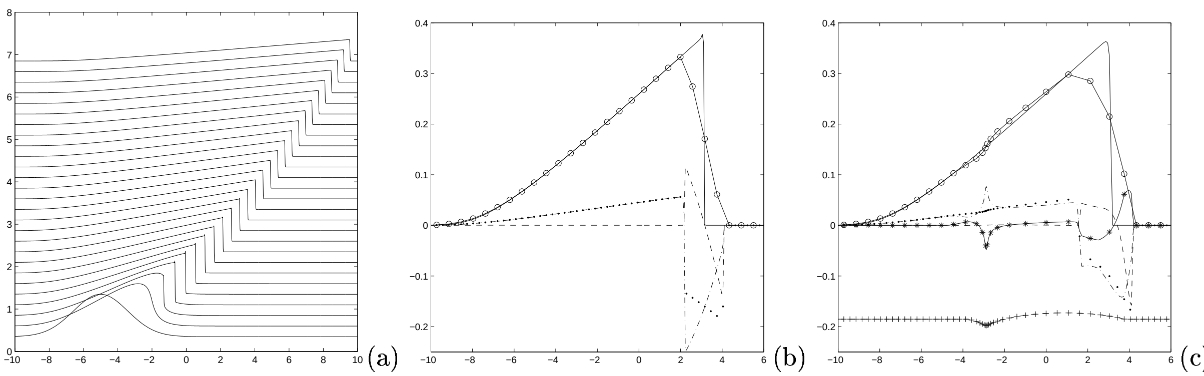

Figure 1: (a): Solution to the Burgers equation initiated with a Gaussian profile, developing into a ramp-cliff structure at $Re = 500$. The solution is shown at times $t = 0, 1, 2, \ldots$. Snapshot of the solution (multiplied by 1/2) (solid) and filtered solution (solid; markers o). Convective flux: total (dots), mean (dash-dotted), turbulent stress (dashed), commutator-error (solid with $*$). In (b) we use $\Delta = \ell/16$ and in (c) a non-uniform case is shown. Underneath in (b), the grid-spacing (minus 0.2) as a function of x is presented.

The Burgers equation provides a model-system which has the same basic structure under filtering. All relevant commutators appear in the filtered Burgers equation. The initial solution is a Gaussian profile which rapidly develops into the well-known 'ramp-cliff' structure. The initial profile has a width equal to 1. We use $Re = 500$ to obtain a sharply localized cliff region. In figure 1(a) we show an example solution. To analyze the LES-fluxes, we collected the contributions to the total convective flux for a representative uniform and non-uniform case. We write

$$\overline{\partial_x(u^2)} = \partial_x(\overline{u}^2) + \partial_x(\overline{u^2} - \overline{u}^2) + \{\overline{\partial_x(u^2)} - \partial_x(\overline{u^2})\}$$

identifying on the right hand side the 'mean', 'SGS' and 'commutator' flux respectively. In figure 1 the solution and the filtered solution are included displaying the 'ramp-cliff' structure. The total flux in figure 1(a) is piecewise linear and the SGS flux is localized in the cliff-region. In figure 1(b) the filtered solution is significantly distorted due to the filter-width non-uniformity. On the 'ramp side', the non-uniform filter-width near $x = -3$ strongly influences the mean flux. The commutator-error compensates for this such that the total flux remains nearly linear in x. The commutator-flux and the SGS-flux have comparable magnitude.