STEEP CAPILLARY WAVES IN ELECTRIFIED FLUID SHEETS

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Summary Nonlinear capillary waves propagating on fluid sheets in the presence of electric fields are considered. Both two-dimensional and axisymmetric configurations are investigated. Numerical solutions are obtained by boundary integral equation methods. In addition analytical aprroximations are calculated by long wave asymptotics. The effects of the electric fields on the wave profiles are studied. New families of capillary waves are presented.

Free liquid films arise in different physical and technological applications in chemical and biomedical engineering, for example. The nonlinear phenomena associated with large interfacial deflections and possible rupture, are of interest in the fundamental understanding of such flows. Physical applications have been discussed by Maldarelli et al (1980) and others.

The linear stability of inviscid free liquid sheets to two-dimensional disturbances was considered by Taylor (1959) who derived dispersion relations for both symmetric (varicose) and antisymmetric (sinuous) modes. Viscosity was introduced by Maldarelli at al (1980) and is found to damp the dispersive waves. In these studies gravity is absent and surface tension is present. In the case of a free surface above an infinitely deep inviscid liquid, exact nonlinear traveling capillary waves were found by Crapper (1957). These solutions were extended to layers of finite thickness by Kinnersley (1976) who constructed nonlinear symmetric waves (for inviscid flows these also describe waves over a horizontal solid surface), as well as nonlinear antisymmetric waves.

We first study the effects of horizontal electric fields on Kinnersley's solutions. The governing equations are

$$\nabla^2\phi(x,y;t) = 0, \tag{1}$$
$$\nabla^2 V^{(1,2)}(x,y;t) = 0, \tag{2}$$

where in what follows superscripts 1, 2 denote the regions bounded by and outside of the moving interfaces (see Figure 1). The electric fields are $\mathbf{E}^{(1,2)} = \nabla V^{(1,2)}$.

The boundary conditions at a free surface $y = S(x,t)$ are the kinematic condition, continuity of normal stresses, and the continuity of the normal displacement and tangential components of the electric field. These are:

$$S_t + \phi_x S_x - \phi_y = 0 \tag{3}$$
$$[\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}]_2^1 = \sigma \mathrm{div}\mathbf{n} \tag{4}$$
$$[\epsilon\mathbf{E}\cdot\mathbf{n}]_2^1 = 0 \tag{5}$$
$$[\mathbf{E}\cdot\mathbf{t}]_2^1 = 0, \tag{6}$$

where $[\cdot]_2^1$ denotes the jump in the quantity as the interface is crossed from the fluid region, the vectors $\mathbf{n}$, $\mathbf{t}$ are the outward pointing normal and tangent to the interface respectively, and the stress tensor $\mathbf{T}$ is given by

$$T_{ij} = -p\delta_{ij} + \mathcal{E}_{ij}, \qquad \mathcal{E}_{ij} = \epsilon\left(E_iE_j - \frac{1}{2}|\mathbf{E}|^2\delta_{ij}\right). \tag{7}$$

The first term in (7) is the inviscid hydrodynamic contribution and the second term arises from interfacial electric stresses given by the Maxwell stress tensor. The parameters ϵ_1, ϵ_2 are the dielectric constants in regions 1 and 2 respectively. The momentum equations can be integrated to yield a Bernoulli equation at the interface. The pressure in region 2 is ambient and equal to a constant and on elimination of the pressure jump across the interface from (4), we arrive at the following Bernoulli boundary condition:

$$\rho_1\left(\phi_t + \frac{1}{2}|\nabla\phi|^2\right) + \frac{1}{1+S_x^2}\left\{S_x^2\left[\mathcal{E}_{11}\right]_2^1 - 2S_x\left[\mathcal{E}_{12}\right]_2^1 + \left[\mathcal{E}_{22}\right]_2^1\right\} = \frac{\sigma S_{xx}}{(1+S_x^2)^{3/2}} + K_p. \tag{8}$$

The Maxwell stresses appearing in (8) are given by

$$\mathcal{E}_{11} = \frac{\epsilon}{2}\left(V_x^2 - V_y^2\right), \qquad \mathcal{E}_{12} = \epsilon V_xV_y, \qquad \mathcal{E}_{22} = \frac{\epsilon}{2}\left(V_y^2 - V_x^2\right), \tag{9}$$

and superscripts 1 and 2 are implied where appropriate in (9).

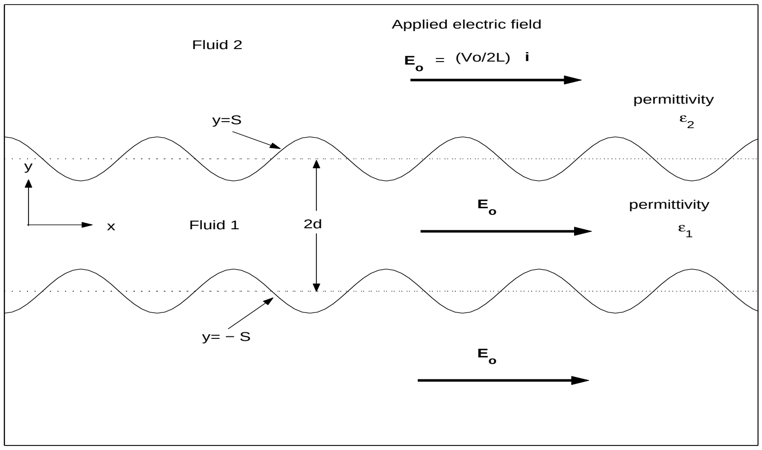

Figure 1: Sketch of the flow configuration. Here the waves are assumed to be symmetric

In the absence of an electric field nonlinear traveling waves (symmetric and antisymmetric) have been constructed analytically by Kinnersley (1976). With electric fields present, we no longer have exact solutions. Therefore we developed weakly nonlinear asymptotic solutions (based on a long wave approximation) and fully nonlinear numerical solutions.

Our numerical method can be described as follows. The three Laplace equations ((1)-(2)) are reformulated as integral equations. This can be done in two ways: (i) using complex variables and Cauchy integral formula, (ii) Green's functions and Green's theorem. Both approaches work equally well in two dimensions, but approach (ii) can be extended to axisymmetric or three-dimensional geometries. The problem is then reduced to one with unknowns on the interfaces. In addition to the shape of the interface, these unknowns are $V^{(1,2)}$ and ϕ. The integral equations are discretized and the resulting nonlinear systems are solved by iteration. The system is more complex than traditional free surface problems due to the additional integral equations and unknowns. The above approach provides a systematic way to derive asymptotic long wave theories. This leads to simpler non-local integro-differential equations which are also addressed numerically. An important finding is that the asymptotic equations give accurate solutions over a long range of system parameters (e.g. imposed electric fields). This is important because the asymptotic models are much faster to solve numerically as compared to the fully nonlinear ones.

Both symmetric and antisymmetric solutions are studied. In addition we show the existence of waves which are neither symmetric or antisymmetric.

The equations (1)-(8) assume a horizontal electric field and non-conducting fluids and neglect the flow in region 2 of Figure 1, for example. Many applications involve vertical electric fields and/or perfectly conducting fluids. For other applications the motion in both regions must be accounted for (in such cases we refer to the free surface as an interface). The last part of the talk will describe some of these extensions.

CONCLUSION

We have developed nonlinear theories to study the effects of electric fields on capillary waves.

References

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