# GENERALIZED INTERNAL SOLITARY WAVES AND FRONTS

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*Summary* It was shown in [1] that waves in a two-layer system with free-surface boundary conditions (or in a three-layer system) can be modelled by a system of two coupled long wave equations. The study of the resonance between a solitary wave of one of the two equations and a copropagating periodic wave of the other equation is carried out numerically. The resulting wave is a generalized solitary wave. It is shown that in the case of a thick solitary wave (solution of a modified Korteweg–de Vries equation with a cubic nonlinearity), the generalized solitary waves do not behave like common $\text{sech}^2$ generalized solitary waves. Simplified models are introduced, which allow a better understanding of these stationary and time-dependent long wave solutions.

## Introduction

This work deals with a particular case of long waves, which arise for example in the physical context of multi-layer flows of inviscid fluids. For instance, two superposed fluids with both a free surface and an interface, or three superposed fluids with two interfaces and a rigid top, are typical configurations of interest here. We are dealing with systems having two wave modes which admit generalized solitary waves. These are nonlinear long wave solutions consisting of a localized central pulse and periodic non decaying oscillations at infinity. These solutions do not only occur in this context. In fact most existing results have been obtained in the framework of the classical water-wave problem with surface tension [2]. It has been proved that, in a specific region of the parameter space, pure solitary waves cannot exist because periodic oscillations, eventually exponentially small, get caught on the solitary pulse.

When dealing with interfacial waves, the amplitude of the solitary pulse can be bounded by the configuration. In the case of classical solitary waves, it is known that when the wave speed approaches a critical value, the solution is best modelled by an extended Korteweg-de Vries (KdV) equation, i.e. a KdV equation with a cubic nonlinear term. The solitary wave reaches its limiting amplitude by infinite broadening. In the limit, one has to consider fronts, which are related to conjugate flows [3]. Some questions arise when dealing with generalized solitary waves: Do oscillations appear on top of the flat part of the wave? Do oscillations only appear on the sides? What are their properties? The objective of this work is to learn more about these new solutions. Rather than proving their existence in a particular context, the idea is to derive models to describe and compare them with common generalized solitary waves.

Dias and Il'ichev [1] derived a system based on two KdV equations with cubic nonlinear and coupling terms. They briefly considered analytically the case with small oscillations on top of a front. A system of two coupled KdV equations was also derived by Grimshaw [4] for three layers with two interfaces, but without emphasis on generalized solitary waves. His point of view consists in treating generically the phenomenon of resonance between two wave modes, in the weakly nonlinear long wave limit. Generalized solitary waves, which are solitary waves with resonant periodic oscillations copropagating with the long wave, appear naturally. Grimshaw's system has been studied by Grimshaw and Iooss [5], who show the existence of such nonlocal solutions. We consider a similar system to study generalized thick solitary waves:

$$
\begin{aligned}
u_t + \Delta u_x + \alpha_1 u u_x + \beta_1 u^2 u_x + \lambda_1 u_{xxx} + \kappa_1 v_x &= 0 \\
v_t - \Delta v_x + \alpha_2 v v_x + \beta_2 v^2 v_x + \lambda_2 v_{xxx} + \kappa_2 u_x &= 0,
\end{aligned}
$$

where $u$ and $v$ are the wave amplitudes. All the coefficients are real.

First we look for stationary solutions. Their properties are discussed and compared with well-known generalized solitary waves. The study of the stationary system provides an approximation for the initial data used when integrating the time-dependent system. The non-stationary solutions are described in terms of the characteristics underlined in the stationary study. In both cases, it is useful to solve analytically a simpler model which clarifies the results.

## Stationary solutions

When the coupling is weak and $v$ of the same order as the coupling, a good approximation of the system is a modified KdV equation in $u$, whose solution forces a linear equation in $v$. This forced equation is certainly the simpler model describing generalized solitary waves. It is completely solvable, so it can be used to get explicit information on the solutions we are interested in. Since we are looking (numerically) for periodic solutions, we choose periodic boundary conditions. Then the domain length is the wavelength of the long periodic wave, which is essentially a solitary wave. In [6], it was shown that the forcing by a flat solitary wave is quite different from the forcing by a $\text{sech}^2$ solitary wave. It is due to the fact that a flat forcing has simple poles outside the imaginary axis in the complex plane. As a consequence, there are isolated parameter values for which the flat solitary wave tends to zero at infinity. If the forcing was the solution of a classical KdV equation, the amplitude of the ripples would never vanish.

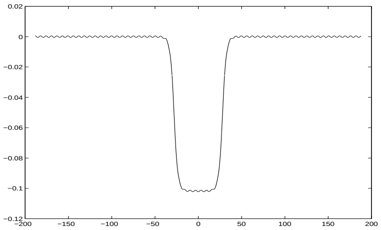

Fig1. *A stationary solution, with ripples on the sides and in the middle part*

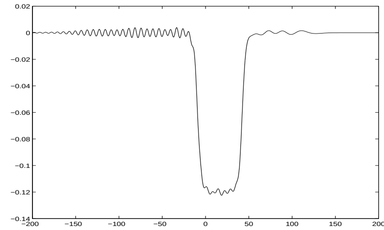

Fig2. *A non-stationary solution*

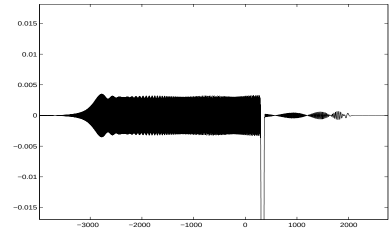

Fig3. *A non-stationary solution after a long time*

Next we compute stationary solutions of the full system numerically (Figure 1). Using spectral methods, we get solutions similar to those of the simplified model. By varying the soliton width, we were able to confirm the previous results on the possibility to kill the ripples.

## Time-dependent solutions

This part deals with the evolution of an initial perturbation. In particular, we want to know if we can get a generalized thick solitary wave. If so, what are its characteristics? As expected, the long wave radiates only on one side as it was discovered for usual generalized solitary waves. The solution after a long time can be compared with stationary solutions of the previous section. On one hand, numerical computations on the whole system are carried out. On the other hand, we use again the forced linear equation ($\alpha_2 = \beta_2 = 0$) to obtain information at very large times.

The numerical scheme combines a spectral method in space with a Runge Kutta time stepping scheme. In order to allow for larger time steps, the integrating factor technique is used. It consists in multiplying each equation by an exponential factor so that the dispersive term no longer appears in the system to be solved. Then the classical fourth order Runge Kutta method is applied (nonlinear terms are computed with the fast Fourier transform and its inverse). The initial perturbation is the thick solitary wave solution of the extended Korteweg-de Vries equation. It gives a good approximation to the core of the expected generalized solitary wave. Indeed, Figure 2 shows such a solution. Note that an initial wave which does not have the proper width splits into several solitary waves. As expected, the ripples radiate only on one side (behind the long wave). We notice the presence of small waves on top of the central pulse. For this moderate time, the wave radiation is not of constant amplitude.

The presence of small oscillations requires small time steps, and so solutions for very long time are difficult to obtain. This is why the simpler model is used again to compare with stationary solutions. This approximation available for weak coupling is solvable in time and leads to an integral formulation of the solution. From this expression, we can plot the solution for any desired time through a numerical evaluation of the integral by fast Fourier transforms. This way, we can check if the stationary solution can be reached. Figure 3 shows a zoom of a solution which underlines the resonance phenomenon. We observe that the envelope modulation is attenuated and slightly oscillates around a constant value. On the other side of the central pulse, there are also some waves which propagate under the form of wave packets. Most likely they correspond to the faster branch of the relation dispersion curve. Their precise origin is still under investigation. If we specify a parameter set such that the stationary solution has vanishing oscillations on the sides as shown in the previous section, we note that the long wave is located just at a trough between two wave packets: therefore the resonance does not occur, or rather occurs with zero amplitude ripples.

## Conclusion

This study is a first step in the study of generalized internal solitary waves and fronts. The time evolution of generalized fronts appears to be quite different from the behaviour of generalized solitary waves.

## References


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