TRANSVERSE INSTABILITY OF SURFACE SOLITARY WAVES

Takeshi KATAOKA , Michihisa TSUTAHARA

Graduate School of Science and Technology, Kobe University, Rokkodai, Nada, Kobe 657-8501, Japan

Summary The linear stability of finite-amplitude surface solitary waves with respect to long-wavelength transverse perturbations is examined by the asymptotic analysis for small wavenumbers of perturbations. The instability criterion is explicitly derived, and it is newly found that there exist transversely unstable surface solitary waves for the amplitude-to-depth ratio of over 0.713. This critical ratio is well below that for the one-dimensional instability (=0.781) obtained by Tanaka [1].

INTRODUCTION

The one-dimensional stability (stability to perturbations that depend only on the main wave's travelling direction) of small-amplitude solitary waves without surface-tension effects was first examined by Jeffrey & Kakutani[2] in the framework of the celebrated Koreweg-de Vries (KdV) equation. Then, these waves were found to be one-dimensionally stable. The transverse stability (stability to perturbations that depend both on the main wave's travelling direction and its transverse direction) was examined first by Kadomtsev & Petviashvili[3]. Then it was found that the small-amplitude solitary waves are also stable to transverse perturbations (see Refs.[4-6]).

The stability of finite-amplitude solitary waves was first examined by Tanaka[1]. He investigated the one-dimensional stability of surface solitary waves numerically, and found that an exchange of stability occurs at their first stationary point of the total energy. The critical amplitude-to-depth ratio is 0.781. Tanaka et al.[4] also conducted numerical simulation of the time development of perturbed solitary waves, and the growth rate of sufficiently small perturbations was found to agree with that of the linear stability analysis.

In contrast, it is only recently that the study on the transverse stability of finite-amplitude solitary waves was initiated. Bridges[7] examined the linear stability of surface solitary waves to long-wavelength transverse perturbations, and found no solitary waves that are transversely unstable. According to his analysis, the surface solitary waves are at the neutral stability to transverse perturbations of small wavenumbers. However, his analysis is based on the leading-order effect of the small wavenumbers only. The higher-order effects therefore, determine the stability.

In the present study, these higher-order effects are investigated. Then, we newly reveal that there exist transversely unstable surface solitary waves. The critical amplitude-to-depth ratio for the transverse stability is found to be 0.713, which is well below that for the one-dimensional stability (=0.781).

LINEAR STABILITY ANALYSIS

Basic equations

Consider the irrotational flow of an incompressible ideal fluid of undisturbed depth D with free surface under the uniform gravitational acceleration g. The set of basic equations is the usual Laplace equation for the velocity potential under the boundary conditions at the bottom (the impermeable condition) and those on the free surface (the kinetic and dynamic conditions). The effects of surface tension are neglected. In what follows, all variables are non-dimensionalized using D and g. Introducing the Cartesian coordinate $x-y-z$ with z pointed vertically upward and its origin placed on the undisturbed free surface, we seek a solution of the following form:

$$\phi = -v\xi + \Phi_s(\xi,z) + \hat{\phi}(\xi,z)\exp(\lambda t + i\varepsilon y), \qquad (\xi = x - vt) \tag{1a}$$

$$\eta = \eta_s(\xi) + \hat{\eta}(\xi)\exp(\lambda t + i\varepsilon y), \tag{1b}$$

where ϕ is the velocity potential and η is the vertical displacement of the free surface. $-v\xi + \Phi_s$ and η_s represent the steady propagation of solitary wave solution at constant speed v in the positive x direction against a uniform flow of constant velocity $-v$. The remaining terms in (1a,b) represent linear perturbations, where ε is a given real constant and λ is a real or a complex constant. The value of λ is determined by solving the following eigenvalue problem for the linear perturbations $\hat{\phi}$ and $\hat{\eta}$, which is obtained by substituting (1a,b) into the basic equations and linearizing with respect to $\hat{\phi}$ and $\hat{\eta}$:

$$\partial^2\hat{\phi}/\partial\xi^2 + \partial^2\hat{\phi}/\partial z^2 = 0, \tag{2a}$$

with the boundary conditions:

$$\partial\hat{\phi}/\partial z = 0 \quad \text{at} \quad z = -1, \tag{2b}$$

$$\left(-\frac{\partial}{\partial z} + \frac{d\eta_s}{d\xi}\frac{\partial}{\partial\xi}\right)\hat{\phi} + \left[\frac{\partial^2\Phi_s}{\partial\xi^2} + \frac{\partial^2\Phi_s}{\partial\xi\partial z}\frac{d\eta_s}{d\xi} + \left(-v + \frac{\partial\Phi_s}{\partial\xi}\right)\frac{d}{d\xi}\right]\hat{\eta} = -\lambda\hat{\eta} \quad \text{at} \quad z = \eta_s, \tag{2c}$$

$$\left[\left(-v + \frac{\partial\Phi_s}{\partial\xi}\right)\frac{\partial}{\partial\xi} + \frac{\partial\Phi_s}{\partial z}\frac{\partial}{\partial z}\right]\hat{\phi} + \left[\left(-v + \frac{\partial\Phi_s}{\partial\xi}\right)\frac{\partial^2\Phi_s}{\partial\xi\partial z} + \frac{\partial\Phi_s}{\partial z}\frac{\partial^2\Phi_s}{\partial z^2} + 1\right]\hat{\eta} = -\lambda\hat{\phi} \quad \text{at} \quad z = \eta_s. \tag{2d}$$

Note that the solitary wave solution is unstable if there exists a localized solution of (2a-d) whose λ possesses the real positive part. In prior study, the stability with respect to perturbations that have no dependence on y (the case of

$\varepsilon = 0$), or the one-dimensional stability was examined numerically by Tanaka[1] as mentioned in the introduction. According to his study, the condition for the solitary waves to be one-dimensionally stable is to satisfy

$$dE/d\mathrm{v} > 0 , \tag{3}$$

where E is the total energy of the solitary wave.

Here we investigate the transverse stability, or the stability with respect to perturbations that depend not only on the ξ direction but also on the y direction, of one-dimensionally stable solitary wave that satisfies the condition (3). For the sake of analytical convenience, we make an asymptotic analysis for small ε to examine the stability to long-wavelength transverse perturbations.

Stability analysis to long-wavelength transverse perturbations

We seek an asymptotic solution of (2a-d) for small ε in the following power series of ε :

$$\hat{\phi} = \hat{\phi}_0 + \varepsilon\hat{\phi}_1 + \varepsilon^2\hat{\phi}_2 + \cdots, \quad \hat{\eta} = \hat{\eta}_0 + \varepsilon\hat{\eta}_1 + \varepsilon^2\hat{\eta}_2 + \cdots, \quad \lambda = \varepsilon\lambda_1 + \varepsilon^2\lambda_2 . \tag{4}$$

Substituting the series (4) into (2a-d) and arranging the same-order terms in ε , we obtain a series of equations for $\hat{\phi}_n$ and $\hat{\eta}_n$ ($n = 0,1,2,\cdots$). At the leading-order (or $n = 0$), the equations are homogeneous and they possess the following leading-order solution: $\hat{\phi}_0 = \partial\Phi_s/\partial\xi$, $\hat{\eta}_0 = d\eta_s/d\xi$.

From the next orders (or $n = 1,2,3,\cdots$), the equations for $\hat{\phi}_n$ and $\hat{\eta}_n$ are inhomogeneous. Since their homogeneous part has the nontrivial solution $(\hat{\phi}_0, \hat{\eta}_0)$, their inhomogeneous terms must satisfy the solvability condition to have a solution. From this condition, the values of λ_1 and λ_2 are determined. Specifically, at $n = 1$, the solvability condition is identically satisfied. At $n = 2$, it becomes

$$\left(\lambda_1^2/\mathrm{v}\right)dE/d\mathrm{v} = -\int_{-\infty}^{\infty}d\xi\int_{-1}^{\eta_s}\left(\partial\Phi_s/\partial\xi\right)^2 dz . \tag{5}$$

This corresponds with the first-order criterion used by Bridges[7]. Recalling that we have assumed that the solitary wave solution satisfies the condition (3), the real part of λ_1 is zero from (5) so that the stability of the solitary wave solution is not determined at this order. To know the stability, we must proceed to the next order.

At $n = 3$, the solvability condition gives

$$\lambda_2 = \begin{cases} \pm Q & \text{if } Q < 0, \\ \text{no solution} & \text{if } Q > 0, \end{cases} \tag{6}$$

where

$$Q = \frac{\mathrm{v}}{2}\left[I + \lambda_1^2\frac{d}{d\mathrm{v}}\left(\frac{I}{\mathrm{v}}\right)\right]\frac{d}{dE}\left(I - \frac{2T}{\mathrm{v}}\right), \tag{7}$$

and I is the impulse, and T is the kinetic energy of the solitary wave. Thus, we have a solution of the eigenvalue problem (2a-d) whose λ possesses the real positive part (or $\lambda_2 = -Q$) when $Q < 0$. That is, the solitary wave solution is unstable to long-wavelength transverse perturbations when $Q < 0$.

Now we apply the instability criterion $Q < 0$ to the solitary wave solution, which is calculated by the numerical method described in Ref.[1]. We found that the function $I + \lambda_1^2 d(I/\mathrm{v})/d\mathrm{v}$ included in Q is always positive. In Fig.1, therefore, $I - 2T/\mathrm{v}$ is plotted as a function of E . If this function takes negative gradient for the one-dimensionally stable range of $dE/d\mathrm{v} > 0$, the corresponding wave is transversely unstable. The circle in the figure represents the point of $dE/d\mathrm{v} = 0$, or the amplitude-to-depth ratio $z_{\max}$ of 0.781, and the solitary wave is one-dimensionally stable for v smaller than that of the circle (or the upper branch from the circle in Fig.1). We see that the function $I - 2T/\mathrm{v}$ takes negative gradient from the cross to the circle. Therefore, there exist solitary waves that are one-dimensionally stable but are transversely unstable in this range. The cross in Fig.1 represents the critical point for the transverse stability, and its amplitude-to-depth ratio $z_{\max}$ is 0.713, which is well below that for the one-dimensional stability $z_{\max} = 0.781$.
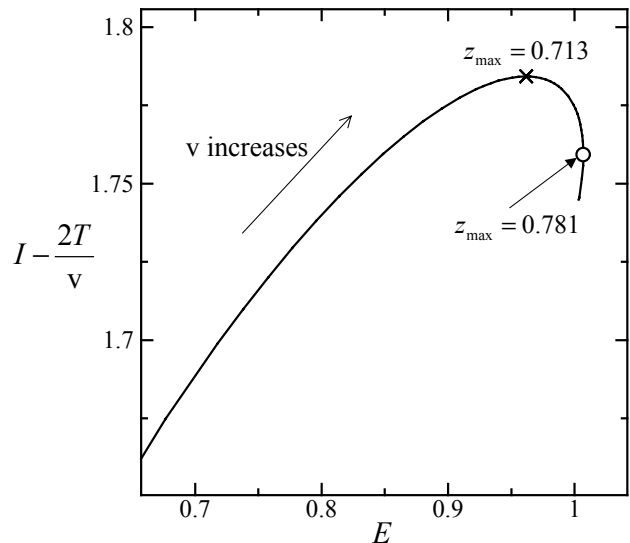

Fig.1 $I - 2T/\mathrm{v}$ as a function of E .

References

[1] Tanaka, M: *Phys. Fluids* **29**:650-655, 1986.
[2] Jeffrey, A., Kakutani, T.: *Indiana Univ. Math.* **20**: 463-468, 1970.
[3] Kadomtsev, B.B., Petviashvili, V.I.: *Sov. Phys. Dokl.* **15**: 539-541, 1970.
[4] Tanaka, M. et al.: *J. Fluid Mech.* **185**, 235-248, 1987.
[5] Zakharov, V.E.: *JETP Lett.* **22**: 172-173, 1975.
[6] Alexander et al.: *Phys. Lett. A* **226**: 187-192, 1997.
[7] Bridges, T.J.: *J. Fluid Mech.* **439**: 255-278, 2001.