

NONLINEAR LONG WAVES ON THE INTERFACE OF A TWO-LAYERED HORIZONTAL FLOW OF VISCOUS LIQUIDS

Dmitry G. Arkhipov*, Georgy A. Khabakhpashev **

*Novosibirsk State University, Pirogov Str. 2, Novosibirsk 630090, Russia

**Institute of Thermophysics SB RAS, Academician Lavrentyev Ave. 1, Novosibirsk 630090, Russia

Summary This paper deals with the theoretical study of plane waves with small but finite amplitude in the two-layer system bounded by the horizontal lid and bottom. The evolution equation for the interface disturbances, which takes into account long-wave contributions of the layers inertia, weakly non-linearity of waves and non-stationary shear stresses at all boundaries of the system is obtained. It is found that amount and direction of a steady unperturbed flow may change not only lengths of disturbances but its polarity too.

In last half a century the dynamics of gravitational waves on surfaces of the liquid shallow currents with shear of longitudinal velocity was studied rather sufficiently (e. g., [1–7]). Recently more attention was paid to such investigations. For instance, model equations for internal waves, which taking into account stationary flows of ideal liquids with piecewise constant dependence of velocity on depth, were suggested in the work [7]. The evolution equation for the interface perturbations in a two-layered system subject to dissipation but failing a steady-state current was deduced in the paper [8]. We note also the analysis [6], where, in particular, it was shown analytically that the two-layered Poiseuille flow is stable in the wide-ranging of the system parameters, and the experiment [5], in which the flow stability for the interface disturbances up to a initiation of a turbulence was demonstrated for the same current of mineral oil and water in the plane one-inch channel. The purposes of the present paper are a derivation of the corresponding model equation which is able to describe a transformation of long nonlinear waves on the interface of two viscous liquids in the presence of a stationary flow in the horizontal channel and an investigation of some its solutions.

It is supposed that two incompressible and immiscible liquids bounded by the rigid fixed lid $z = h_1$ and bottom $z = -h_2$ (unperturbed interface is at $z = 0$), characteristic horizontal lengths of disturbances λ are sufficiently larger and its amplitudes η_a are much less than the layers depths h_l ($h_l/\lambda \sim \varepsilon^{1/2}$ and $\eta_a/h_l \sim \varepsilon$, where ε is a small parameter). Moreover, the capillary effects are assumed to be moderate (the Bond number $Bo = (\rho_2 - \rho_1) g h_1 h_2 / \sigma > 1$, here ρ is the liquid density, g is the acceleration of the free fall, and σ is the surface tension). Finally, the non-stationary laminar boundary layers remain thin, that is the time required for the boundary layer to develop over the entire thickness of the liquid is proposed to be much greater than the characteristic time of the wave process τ (the hydrodynamic homochronicity number $Ho_{\nu l} = \nu_l \tau / h_l^2 \sim \varepsilon^2$, where ν is the kinematical viscosity of the fluid). Below the variable values relating to the stationary flow are marked by the subscript 0 and relating to the interface are marked by the subscript i .

Let c is the phase velocity of linear perturbations in the two-layered system. Than for very long nondissipative waves if $\max u_{0l}(z) < c$ solutions of the motion equations for disturbances of the velocity vertical component have the forms

$$w_{lz} = u_{0i} \frac{b_l}{D} (c - u_{0l}) \frac{4A_l}{\sqrt{D}} \arctan \left((2A_l z + B_l) \frac{u_{0i}}{\sqrt{D}} \right) + u_{0i} \frac{b_l}{D} (2A_l z + B_l) + C_l (c - u_{0l}), \quad (1)$$

here $D = 4A_l u_{0i} (c - u_{0i}) - B_l^2 u_{0i}^2$. We used the kinematical boundary conditions $w_1(h_1) = w_2(-h_2) = 0$ and $w_{lz}(0) = \omega \eta_a (1 - u_{0i}/c)$, where ω is the cyclic frequency of waves. Some these solutions are shown in the Figure 1a. Dissipation affects distinctly on a vertical motion at relatively high velocities of the unperturbed flow (see Figure 1b).

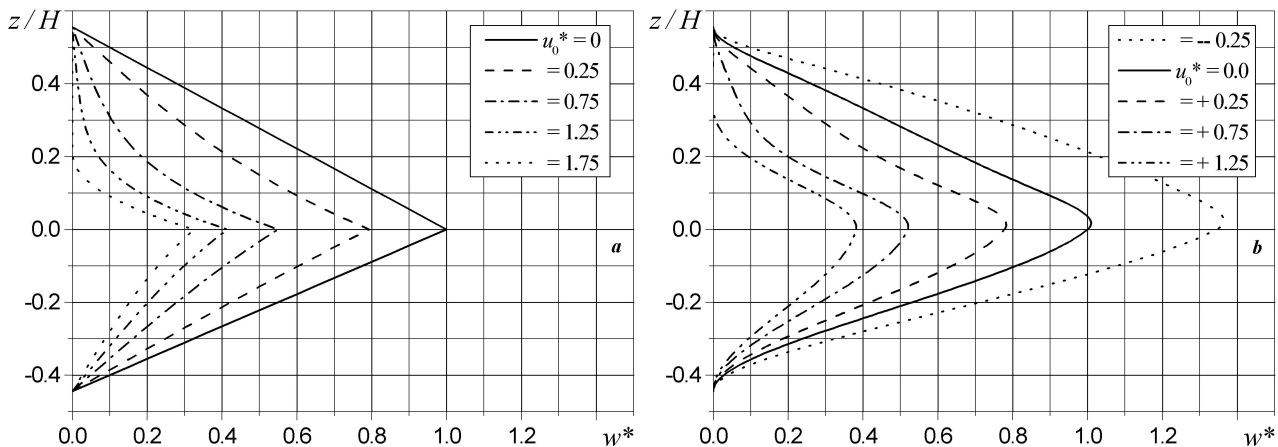


Figure 1. Vertical profiles ($H = h_1 + h_2$) for the dimensionless normal component of the liquid velocity ($w^* = w_z / \omega \eta_a$) calculated by formulas (1) and taking into account a dissipation of linear very long disturbances (cases a and b, respectively) at $\rho_1/\rho_2 = 0.98$, $h_1 = 1.5$ cm, $h_1/h_2 = 1.25$, $\nu_1 = 10^{-4}$ St, $\nu_1/\nu_2 = 0.23$, $\omega = 1$ Hz and different values of the flow velocities ($u_0^* = u_{0i}/c$).

Due to the linear approximations of $w_{lz}(z)$ for small values of the flow velocity, the ordinary boundary conditions and an integration over the z -coordinate the initial system of equations was reduced to following one evolution equation:

$$\frac{\partial^2 \eta}{\partial t^2} + u_{0i}(1 + S_f) \frac{\partial^2 \eta}{\partial t \partial x} - (c_0^2 - u_{0i}^2 S_f) \frac{\partial^2 \eta}{\partial x^2} - C_d \frac{\partial^4 \eta}{\partial t^2 \partial x^2} - C_n \frac{\partial^2 \eta^2}{\partial x^2} + C_b \int_0^t \frac{\partial^2 \eta}{\partial x^2} \frac{dt'}{\sqrt{t-t'}} = 0 \quad (2)$$

Here all the coefficients C_b , C_d , C_n , S_f , and c_0^2 are depended only on the physical (g , ρ_1 , ρ_2 , ν_1 , ν_2 , σ , and u_{0i}) and the geometric (h_1 , h_2) parameters of the system. If we may neglect a dissipation of perturbations ($C_b = 0$) the model equation has steady-state solutions, which contain Jacobi elliptic functions (the periodic conidal and the solitary waves):

$$\eta = \eta_{cn} \text{cn}^2(\xi/L_{cn}, m), \quad \xi = x - Ut,$$

where $L_{cn} = m U \sqrt{6 C_d / (\eta_{cn} C_n)}$, $U = U_{cn} = u_{0i}(1 + S_f)/2 + \sqrt{u_{0i}^2(1 - S_f)^2/4 + c_0^2 + 2 \eta_{cn} C_n(2 - 1/m^2)/3}$;

$$\eta = \frac{\eta_s}{\cosh^2(\xi/L_s)}, \quad L_s = U \sqrt{\frac{6 C_d}{\eta_s C_n}}, \quad U = U_s = \frac{u_{0i}(1 + S_f)}{2} + \sqrt{\frac{u_{0i}^2(1 - S_f)^2}{4} + c_0^2 + \frac{2}{3} \eta_s C_n}.$$

Some periodic and solitary solutions as well as the behavior of the coefficient C_d are represented in the Figure 2. It is shown that amount and direction of an unperturbed flow may change not only lengths of disturbances but its polarity too.

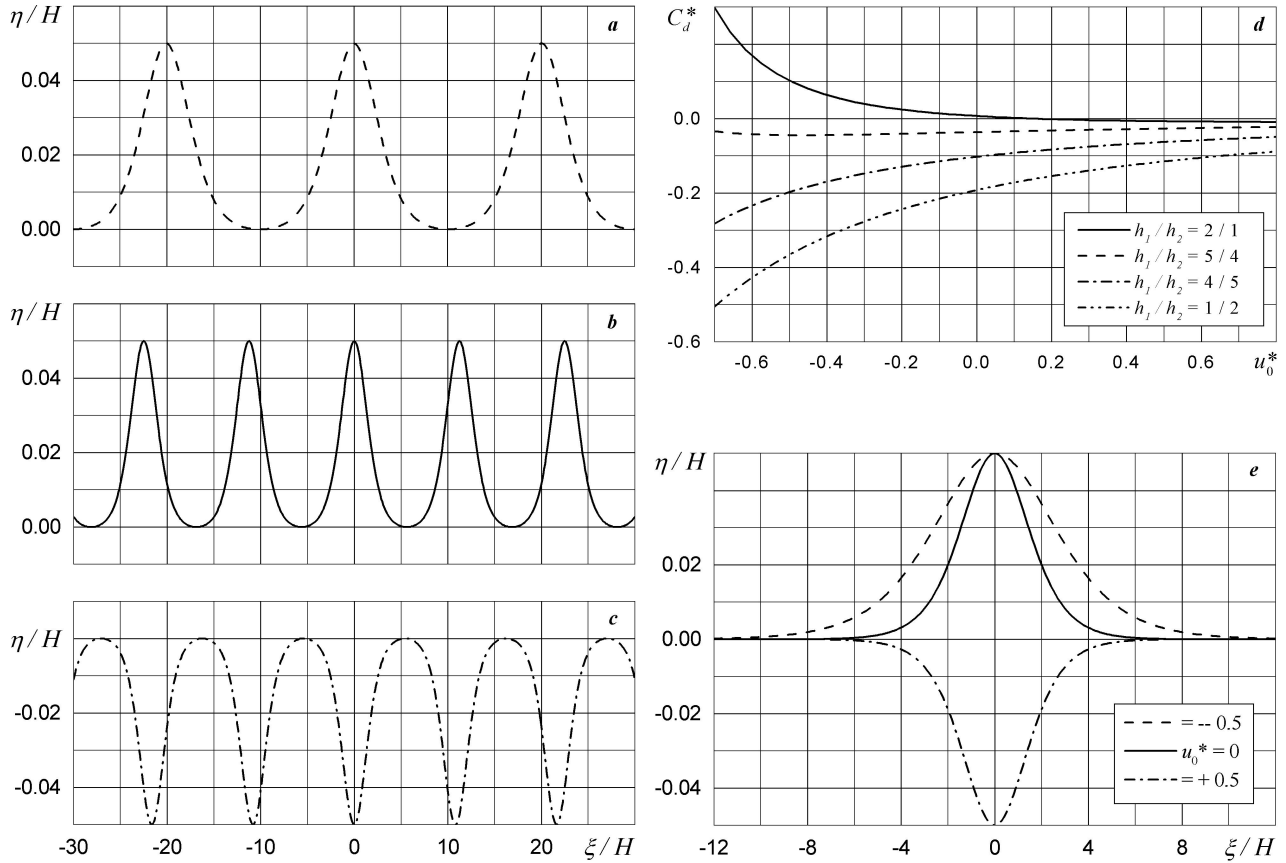


Figure 2. Nonlinear periodical (a, b, c) and solitary (e) waves as well as coefficient C_d (d) at $m = 0.96$, $\eta_{cn}/H = 0.05$ (a, b, c), $\eta_s/H = 0.05$ (e), $\rho_1/\rho_2 = 0.98$, $\nu_1/\nu_2 = 0.23$, $h_1/h_2 = 2$, $h_2 = 2$ cm, $\sigma = 45$ dyn/cm and $u_0^* = -0.5$ (a), 0 (b), 0.5 (c).

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