BIFURCATION OF MOTIONS OF THREE VORTICES AND APPLICATIONS

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Summary With the motion of three point vortices reduced to a problem of two variables, the ratios of the sides of the triangle formed by the vortices, there are stationary points, where the triangles will remain similar. We show the bifurcation from a stationary point of contracting similar triangles ending at the stationary point for expanding triangles. The study is extended to the motion of three coaxial slender vortex rings.

BACKGROUND

The motion of three free point vortices in a planar incompressible inviscid flow was first studied by Groebi in 1877 for specialized vortex strengths and by Synge in 1949 [1], giving a qualitative classifications of all possible motions in terms of the sum of the products of the vortex strengths, $K = k_1k_2 + k_2k_3 + k_3k_1$. It is called elliptic, parabolic or hyperbolic for K greater, equal to or less than 0, respectively. The analysis of Synge was continued in 1988 [2], presenting quantitative results. Figure 1 shows the triangle $\mathcal{G}$ formed by three vortices, at $z_j = Re\ z_j + i\ Im\ z_j$ at instant t, in the complex z-plane, with circulations $2\pi k_j$, and the opposite sides R_j, $j = 1, 2, 3$. With the conservation of linear and angular momentums, the motion was reduced to the variation of the sides of the triangle $\mathcal{G}$, i. e., the trajectory of point $\mathbf{R}(t)$ in space. Synge projected the trajectory radially onto point $\mathbf{x}$ on the $\alpha\beta$ plane, $R_1+R_2+R_3 = \sqrt{2/3}$ and introduced the trilinear coordinates $x_j = R_j/(R_1 + R_2 + R_3)$ as shown in Figure 2. The $\alpha\beta$ plane in the first octant is equilateral triangle $\mathcal{P}$ with vertices P_j of unit heights and the distances of point $\mathbf{x}$ from the sides opposite to P_j, as shown by the dotted line, yield x_j fulfilling the condition, $x_1 + x_2 + x_3 = 1$. Because of the triangle inequality, the point $\mathbf{x}$ has to lie inside or on the boundary of the equilateral triangle, $\mathcal{Q}$, formed by the mid-points Q_j of the sides of $\mathcal{P}$. An vertex of $\mathcal{Q}$, say Q_1 where $R_1 = 0$ or $z_2 = z_1$, represents the coalescence of two vortices, A point on an edge,

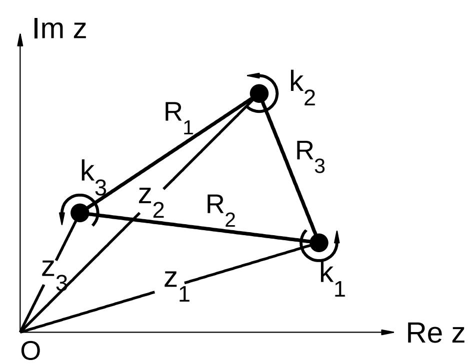

Fig. 1. Configulation of three vortices

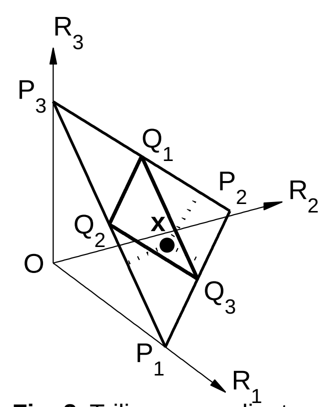

Fig. 2. Trilinear coordinates

say Q_2Q_3 where $x_1 = x_2 + x_3 = 1/2$, the vortices are collinear. Corresponding to a point $\mathbf{R}$, there are two congruent triangles, $\mathcal{G}_\pm$ with the vortices in opposite orientation. Figure 1 shows $\mathcal{G}_+$, with the vortices z_1, z_2, z_3 oriented counterclockwise. We shall set the orientation by assuming $k_1 \geq k_2 = 1 \geq k_3$. When the triangle $\mathcal{Q}$ in the $\alpha\beta$ plane is considered as a disc with faces $\mathcal{Q}_+$ and $\mathcal{Q}_-$, having normal vectors pointing away or toward the origin respectively. Thus a point $\mathbf{x}$ on face $\mathcal{Q}_+$ or or its image on $\mathcal{Q}_-$ represents the trilinear coordinates of the vortices in the counterclockwise or clockwise orientation. The vortices change their orientation only when they cross over an common edge of $\mathcal{Q}_\pm$. When the three vortices move as a rigid body, we have $\mathbf{R}$ stationary or being a critical point. To each configuration defined by a point $\mathbf{R}$ in space there is a unique $\mathbf{x}$ on the $\alpha\beta$ plane. The converse is true for $K \neq 0$. In addition to the vertices of Q_j, there are critical points Q_{j+3} on each edge of $\mathcal{Q}$ and at the centroids $E^\pm$, of $\mathcal{Q}^\pm$. The stabilities of these eight critical points were analyzed in [1] and [2]. The three equations for the variations of $R_j(t)$ are $k_j^{-1}R_j\dot{R}_J = 2A(R_{j+1}^{-2} - R_{j+2}^{-2})$, $j = 1, 2, 3$ with $R_{j+3} = R_j$, γA denoting the area of the triangle $\mathcal{G}$ and $\gamma = \pm 1$ its orientation. They were reduced to two invariants for R_j in [1] defining the integral curves in space. The curves plus the direction of $\dot{\mathbf{R}}(t)$ yield the trajectories. These two invariants were combined to one invariant for the trilinear coordinates [2],

$$I = \{\sum_{j=1}^{3} \frac{x_j^2}{k_j}\}\{x_1^{k_2k_3} x_2^{k_3k_1} x_3^{k_1k_2}\}^{2/K} = \text{const.} \quad K \neq 0 \quad \text{and} \quad I_0 = (\frac{x_1}{x_3})^{2k_1}(\frac{x_2}{x_3})^{2k_2} = \text{const.}\ , \quad K = 0. \qquad (1)$$

When we set the origin of $\alpha\beta$ plane at the vertex Q_3 the α-axes along Q_3P_2 amd β-axis along Q_3P_3, we have $\beta = x_3$ and $\alpha = (x_2 - x_1)/\sqrt{3}$. From the above invariants and the direction of $\dot{\mathbf{x}}$ or $\dot{\mathbf{R}}$, the trajectories of $\mathbf{x}$ in the $\alpha\beta$ plane were presented in [2] for various values of k_j's for the elliptic, parabolic and hyperbolic cases.
For the parabolic case, with $k_3 = -k_1k_2/(k_1 + k_2) < 0$, in addition to the eight critical points $\mathbf{x}$ corresponding to those of $\mathbf{R}$ mentioned before, there are stationary points $\mathbf{x}$ along an arc in $\mathcal{Q}_+$ from critical point Q_4 on the edge Q_2Q_3 passing through the centroid E to point Q_5 on Q_3Q_1, and also on the image of the arc in $\mathcal{Q}_-$. A point $\mathbf{x}$ in the segment Q_4E (EQ_5) represents expanding (contracting) similar triangles $\mathcal{G}_+$. For the image in $\mathcal{Q}_-$, we interchange expanding and contracting for $\mathcal{G}_-$ The contraction of similar triangles to the final collision of three vortices looks like an interesting solution but the solution was shown to be unstable in [2] while the expanding one are stable. Now we shall study the deviation or bifurcation from a contracting similar solution due to instability.

CURRENT RESEARCH

For a parabolic case, the invariant I_0 in (1) decreases from ∞ at the vertex Q_3 where $\beta = x_3 = 0$ to 1 at the centroid E, $\beta = 1/2$ and finally to 0 at the vertex Q_1 or Q_2 where $\beta = 1$. The arc Q_4EQ_5 in $\mathcal{Q}_+$, the loci of stationary points, is defined by $H = k_2x_1^2 + k_1x_2^2 - (k_1 + k_2)\beta^2 = 0$. Crossing the arc from below, H changes sign from negative to positive. Along the arc, $H = 0$, the invariant I_0 decreases from its maximum at E towards its end points. With $k_1 > k_2$, we have $I_0(E) = 1 > I_0(Q_4) > I_0(Q_5)$. The vertex Q_3 is a center and the invariant curves move away from Q_3 as I_0 decreases from ∞ to 1, with the curve $I_0 = 1$ tangent to the arc $H = 0$ at E from below. The trajectory or the separatrix from Q_4 lies above the arc $H = 0$ and ends on the arc at point S_4 between E and Q_5. The separatrix from Q_5 lies above that from Q_4, across over the edge Q_2Q_3 at point S_5 and return to Q_5 along the image of S_5Q_5 in $\mathcal{Q}_-$. The unstable branch EQ_5 is partitioned by point S_4 to two segments. For a point $\mathbf{x}$ in the unstable segment ES_4, the trajectory bifurcates from the point along $I_0 \in (I_0(Q_4), 1)$ above the arc with x_1 increasing while x_2 decreasing, ending at a point in the stable segment Q_4E of the arc representing stable expanding triangle $\mathcal{G}$. For a point $\mathbf{x}$ in the unstable segment S_4Q_5, the trajectory bifurcates from the point along $I_0 \in (I_0(Q_5), I_0(Q_4))$ above the arc with x_2 decreasing to and then crossing over the edge Q_2Q_3 above point Q_4 and returning along the image trajectory to $\mathbf{x}$ in stable segment $S_4^*Q_5$ in $\mathcal{Q}_-$. For a point $\mathbf{x}$ in the entire unstable segment EQ_5, the trajectory bifurcates along $I_0 \in (I_0(Q_5), 1)$ below the arc with x_1 decreasing to and then crossing over the edge Q_3Q_1 to $\mathcal{Q}_-$ and ending at the image of $\mathbf{x}$ in the stable segment E^*S_5. Similarly, we can trace the trajectories bifurcating from the unstable segment Q_4E^* in $\mathcal{Q}_-$ for $I_0 \in (I_0(Q_4), 1)$. Thus completes the description of the trajectories for $I_0 \in (I_0(Q_5), 1)$ in $\mathcal{Q}_\pm$. Note that the centroid E (also E^*) is unstable and looks like a saddle point below the arc $H = 0$ but like a center above.
In addition to the above analyses, bifurcations of three vortex motions are also studied using a Hamiltonian approach thereby providing further insights into the nature of the dynamics. As is well known, the governing equations of motion of three vortices in a plane can be written in a three-degree-of-freedom Hamiltonian form. This system has three independent integrals in involution, including the Hamiltonian function H, and is therefore completely integrable. Two of these integrals can be used to reduce the number of degrees-of-freedom of the system by two. By choosing the reduction properly, this system can be used to obtain the bifurcation results described above and also provide some additional information concerning the motions.
The planar motion of three vortices serves as the leading order approximation of three slender coaxial vortex rings in a meridian plane [3] having radii much larger than the distances between the rings, The approach was used in [4] to analyze certain qualitative features of the motion of three coaxial vortex rings. In particular, they showed, using the KAM and Poincaré-Birkhoff theorems, for vortex strengths having the same sign (the elliptic case), there is a region of initial positions of positive area in a meridian plane for which the relative motion of the rings is quasiperiodic on invariant tori or periodic. A similar formulation shall be employed to investigate the bifurcations of motions of three coaxial vortex rings in the parabolic case.

References

[**1**] Synge, J. I.: On the Motion of Three Vortices. *Can. J. Math.* **1**: 257-270, 1949.
[**2**] Tanvantzis J., Ting L.: The Dynamics of Vortices Revisited. *Phys. Fluids* **31**: 1392-1409, 1988.
[**3**] Ting L., Klein R., Viscous Vortical Flows. Lecture Notes in Physics **374**, Springer-Verlag, NY 1991.
[**4**] Blackmore D., Knio O., KAM Theory Analysis of the Dynamics of Three Coaxial Vortex Rings. *Physica D*, **140**: 321-348 2000.