Helicity Dynamics of Vortex Filaments

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Abstract

We determine the Hamiltonian dynamics in the Lagrangian fluid description describing a massless filament of vorticity $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ supported along a space curve $\mathbf{x} = \mathbf{R}(\mathbf{a}, t)$ that moves without slipping in the incompressible flow induced by its own helicity.

Singular Lagrangian vortex filament solutions induced by the helicity

The Lagrange-to-Euler map for a vorticity filament is, $\boldsymbol{\omega}(\mathbf{x}, t) = \int \frac{1}{J}\,\mathbf{R}_s(\mathbf{a}, t)\delta\big(\mathbf{x}-\mathbf{R}(\mathbf{a}, t)\big)\,d^{\,3}a\,,$ with $J = \det(\partial\mathbf{R}/\partial\mathbf{a})$, and its helicity is,

$$H = \frac{1}{2}\int \mathbf{u}\cdot\operatorname{curl}\mathbf{u}\,d^{\,3}x \;\;=\;\; \frac{1}{8\pi}\int \frac{\mathbf{R}(\mathbf{a},t)-\mathbf{Q}(\mathbf{a}',t)}{|\mathbf{R}(\mathbf{a},t)-\mathbf{Q}(\mathbf{a}',t)|^3}\cdot\frac{d\mathbf{R}}{ds}\times\frac{d\mathbf{Q}}{ds'}\,d^{\,3}a\,d^{\,3}a'\,, \tag{1}$$

as obtained by using the Biot-Savart Law for vorticity filaments,

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi}\nabla\times\int\frac{J\boldsymbol{\omega}(\mathbf{R})\,d^{\,3}R}{|\mathbf{x}-\mathbf{R}|} = \frac{1}{4\pi}\nabla\times\int\frac{\mathbf{R}_s\,d^{\,3}a}{|\mathbf{x}-\mathbf{R}(\mathbf{a},t)|}\,. \tag{2}$$

Here $d/ds' = \boldsymbol{\omega}_{(0)}(\mathbf{a}')\cdot\partial/\partial\mathbf{a}'$ is the derivative along the filament in the primed Lagrangian coordinates and the vectors $\mathbf{R}$ and $\mathbf{Q}$ denote spatial positions either on the same, or different filaments.

The Lagrangian filament dynamics generated by the helicity H in equation (1) may be computed using the Rasetti-Regge Dirac bracket (RRDB) $\{\cdot\,,\,\cdot\}$ derived in [6]. See also [3, 4] for additional discussions and reviews of the RRDB. Hence, the induced Lagrangian filament velocity is found to be,

$$\dot{\mathbf{R}}(s,t) = \{\mathbf{R}\,,\,H\} \;\;=\;\; \frac{1}{|\mathbf{R}_s|^2}\,\mathbf{R}_s\times\frac{\partial}{\partial s}\frac{\delta H}{\delta\mathbf{R}_s} = \frac{-1}{4\pi}\,\frac{1}{|\mathbf{R}_s|^2}\,\mathbf{R}_s\times\frac{\partial}{\partial s}\bigg(\int\frac{\mathbf{R}(s)-\mathbf{R}(s')}{|\mathbf{R}(s)-\mathbf{R}(s')|^3}\times\frac{d\mathbf{R}}{ds'}\,d\,s'\bigg).$$

We apply the local induction approximation (LIA) to the integral in this expression by following the Arms-Hammer procedure [1] as explained, for example, in [5]. Hence, we find the following approximate equation, in which the LIA logarithmic cutoff constant is denoted as $c = \log(L/\sigma)$,

$$\dot{\mathbf{R}}(s,t) = \{\mathbf{R}\,,\,H\} \;\;=\;\; \frac{c}{4\pi}\,\frac{1}{|\mathbf{R}_s|^2}\,\mathbf{R}_s\times\frac{\partial}{\partial s}\Big(\mathbf{R}_s\times\mathbf{R}_{ss}\Big) = \frac{-c}{4\pi}\,\hat{P}\,\mathbf{R}_{sss}\,, \tag{3}$$

with projection operator $\hat{P} = -\hat{\mathbf{t}} \times \hat{\mathbf{t}}\times$, where $\hat{\mathbf{t}} \equiv \mathbf{R}_s/|\mathbf{R}_s|$ is the tangent vector. Equation (3) gives the LIA dynamics of vorticity filament solutions on the space curve $\mathbf{x} = \mathbf{R}(s,t)$ obtained by using the helicity (1) as the Hamiltonian H in the RRDB. When the parameter s is transformed into arclength along the curve, by $d\ell = |\mathbf{R}_s|ds$, so that $\partial_s = |\mathbf{R}_s|\partial_\ell$, equation (3) may be put into "intrinsic" form,

$$\dot{\mathbf{R}}(s,t) \;=\; \frac{-c}{4\pi}\,\hat{P}\,\mathbf{R}_{sss} = \frac{-c}{4\pi}\left(\tilde{\kappa}\,'\hat{\mathbf{n}} + \tau\tilde{\kappa}\,\hat{\mathbf{b}}\right). \tag{4}$$

Here $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ are the Serret-Frenet unit vectors for the space curve, and τ and κ are torsion and curvature, respectively. We define $\tilde{\kappa} \equiv \kappa|\mathbf{R}_s|^2$ and $\tilde{\kappa}\,' \equiv d\tilde{\kappa}/d\ell$. Note that $\mathbf{R}_s \cdot \dot{\mathbf{R}}_s = -\dot{\mathbf{R}} \cdot \mathbf{R}_{ss} = (c\,/4\pi)\,\tilde{\kappa}\,'\kappa \neq 0$. Hence, equation (4) allows self stretching by an isolated filament with curvature, so that $|\mathbf{R}_s|^2$ does not remain constant, even in the LIA. We compare this with the da Rios-Betchov equation, [3, 7, 8]

$$\dot{\mathbf{R}}(s,t) = (1/|\mathbf{R}_s|^2)\,(\mathbf{R}_s \times \hat{\mathbf{t}}_s) = \kappa\,\hat{\mathbf{b}}\,.$$

This is generated by RRDB with Hamiltonian $H_{dRB} = \int |\mathbf{R}_s|ds$ and it does not allow self stretching.

Outlook We have determined the Hamiltonian dynamics for the singular solutions generated by the helicity invariant of the divergenceless vorticity vector field. Conservation laws and other natural directions remain to be explored. For example, one should investigate whether the Hasimoto transformation [2] for Kelvin-wave solitons on vortex filaments also leads to interesting phenomena in the present situation. In particular, applying the Hasimoto transformation to the intrinsic equation (4) for the Hamiltonian motion of vortex filaments induced by the helicity (1) leads to yet another generalization of the nonlinear Schrödinger equation which may have interesting dynamics.

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