MIXING IN MULTICONNECTED PLANAR DOMAINS

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Summary We study the stirring of two-dimensional fluids in multi-connected domains with moving boundaries using analytic and numerical tools. For time-periodic Euler flows, the classical Helmholtz-Kelvin Theorem shows that for typical initial vorticity such flows are not chaotic. On the other hand, it is shown that for constant initial vorticity there are stirring protocols which always yield chaotic time-periodic Euler flows. These protocols are those that generate flow maps in pseudoAnosov isotopy classes. These classes are a basic ingredient of the Thurston-Nielsen theory and a further application of that theory shows that pseudoAnosov stirring protocols with generic initial vorticity always yield solutions to Euler's equations for which the sup norm of the gradient of the vorticity grows exponentially in time.

We also investigate Stoke's flow with the same stirring geometry. Numerical and experimental results have shown that the flow maps under pseudoAnosv protocols appear to lack elliptic islands on a visible scale and thus are very thorough stirring procedures. Our investigations center on the connection between the classical Helmhotz characterization of Stoke's flow as minimizing the L^2-norm of the deformation tensor and the quasiconformal distortion of the flow map. Minimization of the latter under the appropriate conditions is known to imply global hyperbolicity.

INTRODUCTION

We study the stirring of two-dimensional fluids in multi-connected regions with moving boundaries using analytic and numerical tools. The outer boundary is fixed and the inner boundaries or "strirrers" are moved in a T-periodic protocol. We usually restrict to the case that the resulting velocity field is also periodic and study the time T-flow (or Poincaré) map f. As has been well established, the methods of dynamical systems theory can be used to analyze the map f and thus gain basic insights into the kinematic template underlying the fluid stirring.

Our analysis centers on the way in which the topology of the motion of the stirrers influences the stirring of the flow map. This topology is described by assigning a braid to each stirring protocol by visualizing its motion in three dimensional space-time. This braid determines the isotopy class of the flow map and the isotopy class information is then used in the Thurston-Nielsen Theory. In the case of stirring protocols of pseudoAnosov type, this theory provides *a priori* lower bounds on such quantities as the stretching of material lines and the topological entropy. These bounds depend only on the braid of the stirring protocol and not the details of the stirrer motion, and further, are valid for any fluid model, eg. ideal flow or Stokes flow, as long as the fluid region is maintained as connected continuum.

This circle of ideas was introduced in [2] and has been studied in [4] and [5]. In this paper we examine the implications of the stirring topology on the dynamics of the flow map under the assumption of a specific fluid model. We restrict attention here to incompressible, constant density flow and examine the two extreme cases of inviscid (Euler) flow and quasi-stationary Stokes flow.

EULER FLOW

We first investigate the dynamics of time-periodic Euler flows in multi-connected, planar fluid regions which are "stirred" by the moving boundaries. The classical Helmholtz-Kelvin theorem on the transport of vorticity implies that if the initial vorticity of such a flow is generic among real-valued functions in the C^k-topology ($k \geq 2$) or is C^ω and nonconstant, then the flow has zero topological entropy (the C^ω case is equivalent to Remark 4 in [3]).

Amongst the Euler flows with non-generic vorticity those with constant vorticity are of particular interest. Using standard potential theory, a proposition shows that systems with constant vorticity and periodic stirring protocols always give rise to periodic Euler fluid motions. Further, if the stirring protocol is of pseudoAnosov type, then the resulting flow maps always have positive topological entropy and thus are chaotic. Since conditions that ensure periodic solutions to the Euler equations are very rare, this result is useful in guaranteeing the existence of at least one interesting class of time T-Euler flow maps.

Using these results in the real analytic case yields a dichotomy that is somewhat similar to that of Arnol'd for steady 3D Euler flows. For 2D time-periodic Euler flows, if the vorticity is nonconstant, then the dynamics are "integrable" and the entropy is zero. One may have chaotic dynamics in the constant vorticity case, but only if the fluid is stirred.

As another consequence of the Thurston-Nielsen theory of surface automorphisms, we also show that for generic initial vorticity there are large classes of periodic stirring protocols which never give rise to periodic Euler fluid motions. Further, these stirring protocols result in exponential growth of the sup norm of the gradient of the vorticity. As a consequence of the preservation of vorticity one expects this general type of behavior in any chaotic Euler flow with generic smooth vorticity. The attractive feature of the theorem is that the topology of the stirrer motion allows one to get concrete results on the exponential growth.

[4]

QUASI-STATIONARY STOKES FLOW

One of the striking feature of the viscous flow maps reported from experiment in [2] and numerical computations in [4] and [5] is the apparent lack of elliptic islands. Elliptic islands are regions of the fluid associated with periodic points whose multipliers are on the unit circle. Since the region enclosed by the invariant circles is invariant under the action of the flow map, elliptic islands give patches of fluid that do not mix with the surrounding fluid. The apparent lack of elliptic islands thus indicates a highly thorough stirring protocol.

We present an analytic and numerical investigation of this phenomenon using quasi-Stationary Stokes flow as a model. We give evidence that the lack of elliptic islands can be explained using the classic Helmholtz variational principle in which Stoke's flow is characterized amongst divergence-free vector fields with given boundary values as minimizing the rate of energy dissipation as measured by the L^2-norm of the deformation tensor. This property is, in turn, connected to properties of the flow map, with particular importance given to the quasi-conformal (qc) distortion. The qc-distortion measures how much infinitesimal circles are distorted by the map, and it is, in many ways, the analog for maps of the rate of energy dissipation for a velocity field. It is known that the global minimization (in the appropriate class of maps and metrics) of the qc-distortion is connected to hyperbolicity and in fact can be used to characterize pseudoAnosov maps ([1]). While we have not been able to make these connections rigorous, and in fact have counterexamples to various conjectured connections. we do have numerical evidence for the connection of the two measures of distortion on certain "scales" and use it to provide a plausible explanation for the lack of elliptic islands above a certain length scale.

References

[1] Bers, L., An extremal problem for quasiconformal mappings, *Acta. Math.*, **141**, 1978, 73–98.
[2] Boyland, P., Aref, H. and Stremler, M., Topological fluid mechanics of stirring, *J. Fluid Mech.*, **403**, 277–304, 2000.
[3] Brown, M. & Samelson, R., Particle motion in vorticity- conserving, two-dimensional incompressible flows, *Phys. Fluids*, **6**, 2875–2876, 1994.
[4] Finn, M., Cox, S. & Byrne, H., Topological chaos in inviscid and viscous mixers, *J. Fluid Mech.*, **493**, 345–361, 2003.
[5] Vikhansky, A., Simulation of topological chaos in laminar flow, *Chaos*, 2004 (in press).